Equivalency of Lightning Return-Stroke Models Employing Lumped and Distributed Current Sources

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***Abstract*—We show that any engineering return-stroke model can be expressed, using an appropriate continuity equation, in terms of either lumped or distributed current sources with the resultant longitudinal-current distribution along the channel being the same. This property can be viewed as the duality of engineering models. The conversion alters the actual-corona current (if any) of the model. For lumped-source (LS) models the actual-corona current is unipolar and directed radially out of the channel core, while for distributed-source (DS) models it is unipolar and directed into the channel core. For LS models converted to DS models and for the Diendorfer–Uman (DU) model converted to the equivalent LS model, the corona current is the sum of the negated actual-corona current (if any) and a fictitious-corona current, the latter being bipolar. For the transmission-line (TL) model (no longitudinal current attenuation with height) expressed in terms of DSs, there is only a fictitious bipolar corona current component. Conversion of the traveling-current source (TCS) and Bruce–Golde (BG) models to equivalent LS models involves replacement of the actual, unipolar corona current with a fictitious one, the latter current being bipolar near the channel base and unipolar at higher altitudes.**

***Index Terms*—Corona sheath, current continuity equation, distributed sources (DSs), duality, lightning, lumped source (LS), return-stroke model.**

I. Introduction

COORAY [1] showed that any engineering model implying a lumped current source at the lightning-channel base (any transmission-line (TL)-type model) can be formulated in terms of sources distributed along the channel and progressively activated by the upward-moving return-stroke front. This has been previously demonstrated for one model (modified TL model with exponential-current decay with height) by Rachidi and Nucci [2]. An engineering return-stroke model is defined here as an equation relating the longitudinal channel current (at any height and any time) to the current at the channel origin (or an equivalent equation in terms of the line charge density) [3], [4]. The approach suggested by Cooray [1] was used by Rachidi *et al.* [5] to generalize five engineering models in order to take into account a tall strike object.

Conversion of a lumped-source (LS) model to an equivalent distributed-source (DS) model, both illustrated in Fig. 1, involves the use of 1) the continuity equation derived for a downward-propagating current wave and 2) the longitudinal-current equation for the LS model. We will show in this paper that a DS model can be converted to equivalent LS model using 1) the continuity equation derived for an upward-propagating current wave and 2) the longitudinal-current equation for the DS model. Longitudinal current equations for both LS and DS models, often referred to as TL type and traveling current-source type models, respectively, are summarized by Rakov and Uman [3]. Note that in DS models [see Fig. 1(b)], corona current is injected into the channel core to drain the charge stored around the core by preceding leader. On the other hand, in LS models [see Fig. 1(a)] corona current flows out of the channel core to neutralize the charge deposited around the core by preceding leader.

Although the direction of propagation of the longitudinal current wave is upward for the LS models and downward for the DS models, the charge of the same sign is transported to ground in both types of models. The overall longitudinal current distribution along the channel extends upward for any model, as illustrated in [4, Fig. 12.8].

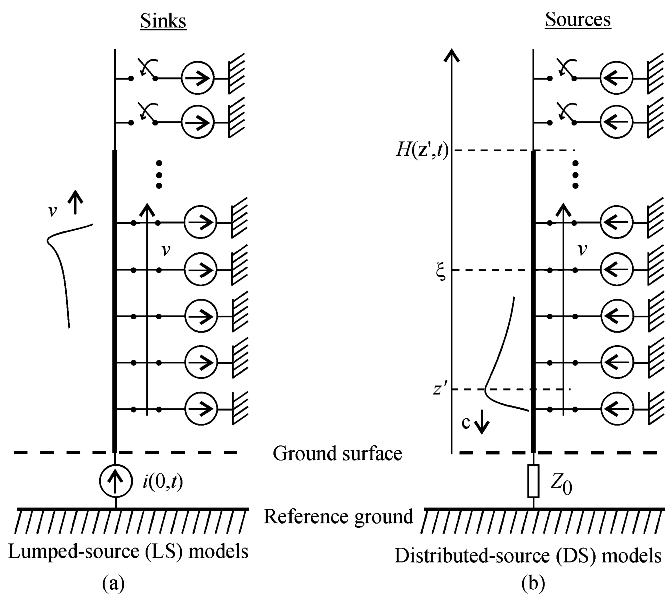

Fig. 1. Schematic representation of engineering return-stroke models that employ (a) a lumped current source at the lightning channel base (LS models) and (b) distributed current sources along the channel (DS models). v is the upward return-stroke front speed, c is the speed of light, and Z_0 is the characteristic impedance of the lightning channel (matched conditions at ground are implied in DS models). LS models with longitudinal-current decay with height imply current sinks distributed along the channel, as shown in (a).

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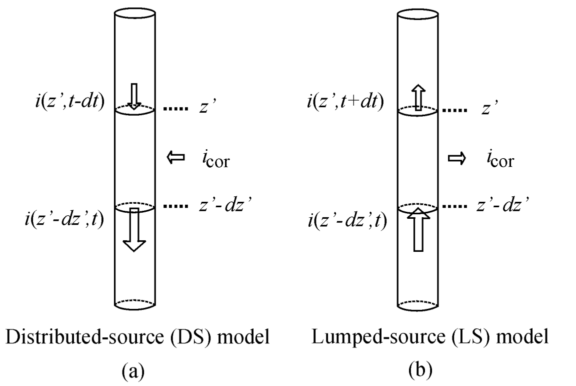

Fig. 2. Differential channel segments used in deriving continuity equations for (a) DS models and (b) LS models. i and $i_{\rm cor}$ are the longitudinal and radial corona currents, respectively.

We will also show that the conversion alters the actual corona current of the model. Specifically, for LS models that are converted to DS models, the corona current is the sum of the negated actual current and a fictitious current, the latter being bipolar. For the TL model expressed in terms of DSs, there is only fictitious corona-current component, since no actual corona current can flow in this model.

II. Continuity Equation for DS Models and Conversion of LS Models to DS Models

Cooray [1], considering a differential channel segment [see Fig. 2(a)] and using Kirchhoff's current law, derived the following continuity equation for DS models:

$$\frac{i_{\rm cor}(z',t)}{dz'} = i'_{\rm cor}(z',t) = -\frac{\partial i(z',t)}{\partial z'} + \frac{1}{c}\frac{\partial i(z',t)}{\partial t} \quad (1)$$

where $i(z',t)$ is the longitudinal channel current at height z' and time t, c is the speed of light, $i_{\rm cor}(z',t)$ is the radial corona current at height z' and time t, and $i'_{\rm cor}(z',t)$ is the radial corona current per unit length injected into the channel.

Note that the corona current given by (1) is the actual corona current only for DS models where the current wave propagates downward at the speed of light c. The most used models of this type include the Diendorfer–Uman (DU) and traveling-current source (TCS) models. The Bruce–Golde (BG) model [6] is often also assigned to this type. For the DU model [7], the radial-corona current per unit channel length, derived from (1), is given by

$$i'_{\rm cor}(z',t) = e^{-(t-z'/v)/\tau_D} \times \left[\left.\frac{di(0,t)}{v^* dt}\right|_{t=z'/v^*} + \frac{i(0,z'/v^*)}{v^*\tau_D} \right], \quad t \ge z'/v \quad (2)$$

where $v^* = (1/v + 1/c)^{-1}$, and τ_D is the discharge time constant. This corona current represents progressively activated sources that inject into the channel core the charge deposited around the core by preceding leader. For the TCS model [8], $i'_{\rm cor}(z',t)$ can be expressed, based on (1), as

$$i'_{\rm cor}(z',t) = \frac{1}{v^*} i\left(0, \frac{z'}{v^*}\right) \delta(t - z'/v) \quad (3)$$

where $\delta(t - z'/v)$ is the Dirac delta function. For the BG model, the corona-current equation can be obtained from (3) by replacing c with ∞ in the expression for v^*, such that $v^* = v$. Equations (2) and (3) give actual corona currents for the DU and TCS models, respectively. These currents are unipolar and are directed radially into the channel core.

One can obtain the charge per unit channel length $\rho(z',t)$ for the DU and TCS models by substituting (2) and (3) in the following equation:

$$\rho(z',t) = -\frac{i(z',t)}{c} + \int_{z'/v}^{t} i'_{\rm cor}(z',\tau)\, d\tau, \quad t \ge z'/v \quad (4)$$

Equation (4) is another form of the continuity equation for DS models, which can be readily derived from (1).

One can formally define the equivalent corona current for LS models, using continuity equation (1), as done in [1] and [5]. The most used LS models include the TL model and modified TL models with linear (MTLL) and modified TL models with exponential (MTLE) current decay with height [9]–[11]. For these models, (1) and (4), for $t \ge z'/v$, become as follows:

TL Model:

$$i'_{\rm cor}(z',t) = \frac{1}{v^*}\frac{\partial i(0,t-z'/v)}{\partial t}$$
$$\rho(z',t) = \frac{i(0,t-z'/v)}{v} \quad (5a)$$

MTLL Model:

$$i'_{\rm cor}(z',t) = \frac{1}{v^*}\frac{\partial i(0,t-z'/v)}{\partial t}\left(1 - \frac{z'}{H}\right) + \frac{i(0,t-z'/v)}{H}$$
$$\rho(z',t) = \frac{i(0,t-z'/v)}{v}\left(1 - \frac{z'}{H}\right) + \frac{1}{H}\int_{z'/v}^{t} i(0,\tau - z'/v)d\tau \quad (5b)$$

MTLE Model:

$$i'_{\rm cor}(z',t) = \frac{1}{v^*}\frac{\partial i(0,t-z'/v)}{\partial t} e^{-z'/\lambda} + \frac{e^{-z'/\lambda}}{\lambda} i(0,t-z'/v)$$
$$\rho(z',t) = \frac{i(0,t-z'/v)}{v} e^{-z'/\lambda} + \frac{e^{-z'/\lambda}}{\lambda}\int_{z'/v}^{t} i(0,\tau - z'/v)d\tau \quad (5c)$$

where H is the total channel length, and λ is the current-decay height constant. We will show in Section V that the corona current for LS models converted to equivalent DS models is the sum of the negative of the actual corona current and a fictitious corona current.

Equations (5a)–(5c) can be used to express the longitudinal current for LS models in terms of DSs as [2], [5]

$$i(z',t) = \int_{z'}^{H} i'_{\rm cor}\left(\xi, t - \frac{\xi - z'}{c}\right) d\xi \qquad (6)$$

where z' denotes the height below the return-stroke front, $H = H(z',t) = v^*(t + z'/c)$ is the return-stroke wave front height as seen by observer at height z', $v^* = (1/v + 1/c)^{-1}$, and ξ is the height between z' and H [see Fig. 1(b)]. Note that (6) can be also used to express the longitudinal current for the DU and TCS models in terms of the corona current $i'_{\rm cor}(z',t)$ per unit channel length, which is given by (2) and (3), respectively.

III. Continuity Equation for LS Models

As stated in Section I, one can also convert DS models to equivalent LS models. In order to do this, we first derive the continuity equation for LS models, which relates the longitudinal return-stroke current to the radial-corona current per unit channel length. Such continuity equation for LS models is presently not found in the literature. Two different approaches can be employed: 1) using the continuity equation formulated in [12] as the starting point for LS models and 2) considering a differential channel segment and using Kirchhoff's current law, which is an approach similar to that used in [1] in deriving the continuity equation for DS models (see Section II). Using these two methods, we will derive two different but equivalent continuity equations for LS models, which will be combined to obtain an expression for the actual corona current in LS models. The latter expression, in turn, will be used in deriving a charge-density equation in terms of both longitudinal and corona currents for LS models.

A. Derivation of Continuity Equation for LS Models Using Thottappillil et al. [12, eq. (20)]

The general continuity equation derived for the return stroke by Thottappillil *et al.* [12] relates the charge density per unit channel length at any height z' and time t to the longitudinal return-stroke current

$$\rho(z',t) = \frac{i(z', z'/v)}{v} - \int_{z'/v}^{t} \frac{\partial i(z',\tau)}{\partial z'}\, d\tau. \qquad (7)$$

The longitudinal-current distribution along the channel for LS models is specified by Rakov and Uman [3] as

$$i(z',t) = P(z')\, i(0, t - z'/v)\, u(t - z'/v) \qquad (8)$$

where $i(0,t)$ is the channel-base current, $P(z')$ is the height-dependent current attenuation factor introduced by Rakov and Dulzon [13], and $u(t - z'/v)$ is the unit-step function. We can rewrite equation (7) using (8) as

$$\rho(z',t) = \frac{i(z',t)}{v} - \frac{dP(z')}{dz'} \int_{z'/v}^{t} i(0, \tau - z'/v)\, d\tau. \qquad (9)$$

Taking the time derivative on both sides of (9), we get

$$\frac{\partial \rho(z',t)}{\partial t} = \frac{1}{v}\frac{\partial i(z',t)}{\partial t} - \frac{dP(z')}{dz'} i(0, t - z'/v). \qquad (10)$$

Below the propagating return-stroke front, that is, for $t > z'/v$, the continuity equation in a general form can be expressed as [12]

$$\frac{\partial \rho(z',t)}{\partial t} = -\frac{\partial i(z',t)}{\partial z'}. \qquad (11)$$

Substituting (11) in (10), we can write

$$-\frac{dP(z')}{dz'} i(0, t - z'/v) = -\frac{\partial i(z',t)}{\partial z'} - \frac{1}{v}\frac{\partial i(z',t)}{\partial t}. \qquad (12)$$

Equation (12), which is equivalent to (9) and therefore represents the continuity equation for LS models, will be employed below to obtain an equation for the actual corona current in LS models.

B. Derivation of Continuity Equation for LS Models Using the Kirchhoff's Current Law

Consider a lightning return-stroke current wave that propagates upward from ground to the cloud [see Fig. 1(a)]. As the current injected at the ground surface traverses the channel segment shown in Fig. 2(b), the radial-corona current in general will cause a reduction of this current. As a result, current at the top of the channel segment will be smaller than that at its bottom. The difference in these two currents will give the total corona current flowing radially outward from the channel segment. For channel segment dz' located at height z' above ground, we can write

$$i_{\rm cor}(z',t) = i(z' - dz', t) - i(z', t + dt) \qquad (13)$$

where $i(z' - dz', t)$ and $i(z', t + dt)$ are the currents at the bottom and top of the channel segment, respectively, and $i_{\rm cor}$ is the radial-corona current which serves to neutralize the usually negative charge deposited along the channel by preceding leader. Using the Taylor's expansion, we can rewrite (13) as

$$i_{\rm cor}(z',t) = i(z' - dz', t) - \left[i(z',t) + dt\frac{\partial i(z',t)}{\partial t}\right]. \qquad (14)$$

Taking into account the fact that $dt = dz'/v$

$$i_{\rm cor}(z',t) = i(z' - dz', t) - i(z',t) - \frac{dz'}{v}\frac{\partial i(z',t)}{\partial t}. \qquad (15)$$

After dividing both sides of (15) by dz', we find

$$\frac{i_{\rm cor}(z',t)}{dz'} = i'_{\rm cor}(z',t) = -\frac{i(z',t) - i(z' - dz', t)}{dz'} - \frac{1}{v}\frac{\partial i(z',t)}{\partial t}. \qquad (16)$$

For $dz' \to 0$, the corona current per unit length $i'_{\rm cor}$ "bleeding off" the lightning channel (this current flows in the radial-corona sheath surrounding the channel core which carries the longitudinal current) at height z' is given by

$$i'_{\rm cor}(z',t) = -\frac{\partial i(z',t)}{\partial z'} - \frac{1}{v}\frac{\partial i(z',t)}{\partial t}. \qquad (17)$$

IV. Conversion of DS Models to LS Models

We now combine the result of Section III-B with that obtained in Section III-A. Since the right-hand sides of (12) and (17) are identical, their left-hand sides should be equal as well and the resultant equation is given by

$$i'_{\rm cor}(z',t) = -\frac{dP(z')}{dz'} i(0, t - z'/v). \tag{18}$$

Equation (18) describes the actual radial corona current for LS models, because this current is associated with the deposited charge-density component introduced by Thottappillil *et al*. [12]. Indeed, taking the time integral of both sides of (18) from $-\infty$ to t, we get

$$\int_{z'/v}^{t} i'_{\rm cor}(z',\tau)\,d\tau = -\frac{dP(z')}{dz'}\int_{z'/v}^{t} i(0,\tau - z'/v)\,d\tau \tag{19}$$

where the right-hand side is the same as the second term given in [12, eq. (20)] and defined as the deposited charge-density component.

Note that (17), which is the continuity equation derived for LS models (the longitudinal-current wave propagating upward), is different from (1), which is the continuity equation derived by Cooray [1] for DS models (the longitudinal-current wave propagating downward).

Equation (9), which is the continuity equation derived for LS models [12], is equivalent to (12) and can be rewritten using (18) as

$$\rho(z',t) = \frac{1}{v} i(z',t) + \int_{z'/v}^{t} i'_{\rm cor}(z',\tau)\,d\tau \tag{20}$$

Note that the second term of (20) represents the deposited charge-density component in terms of the corona current per unit channel length. This equation is the LS-model counterpart of (4), the latter being derived for DS models.

In the following, we will obtain equations for the corona current per unit length and charge per unit length for DS models converted to equivalent LS models using (17) and (20), respectively.

A. DU Model

The longitudinal-current equation for the DU model is [7]

$$i(z',t) = \left[i(0,t+z'/c) - i(0,z'/v^*)e^{-(t-z'/v)/\tau_D}\right] \times u(t - z'/v). \tag{21}$$

After substituting (21) in (17) and applying some algebra, we can write

$$\begin{aligned}
i'_{\rm cor}(z',t) &= -\frac{\partial}{\partial z'}\left\{\left[i(0,t+z'/c) - i(0,z'/v^*)e^{-(t-z'/v)/\tau_D}\right]u(t-z'/v)\right\}\\
&\quad - \frac{1}{v}\frac{\partial}{\partial t}\left\{\left[i(0,t+z'/c) - i(0,z'/v^*)e^{-(t-z'/v)/\tau_D}\right]u(t-z'/v)\right\}\\
&= -\frac{1}{v^*}\frac{\partial i(0,t+z'/c)}{\partial t} + e^{-(t-z'/v)/\tau_D}\left.\frac{di(0,t)}{v^* dt}\right|_{t=z'/v^*}, \quad t \ge z'/v.
\end{aligned} \tag{22}$$

We can rewrite (22) in an equivalent form, which, similar to LS models converted to DS models (see Section V), contains the negative of actual corona current. In order to show this, we expand the first term of (22) as

$$\begin{aligned}
-\frac{1}{v^*}\frac{\partial i(0,t+z'/c)}{\partial t} &= -\frac{1}{v^*}\frac{\partial}{\partial t}\left[i(0,t+z'/c) - i(0,z'/v^*)e^{-(t-z'/v)/\tau_D}\right]\\
&\quad + \frac{i(0,z'/v^*)}{v^*\tau_D} e^{-(t-z'/v)/\tau_D}.
\end{aligned}$$

Substituting the right-hand side of this equation in (22), we get

$$\begin{aligned}
i'_{\rm cor}(z',t) &= -\frac{1}{v^*}\frac{\partial}{\partial t}\left[i(0,t+z'/c) - i(0,z'/v^*)e^{-(t-z'/v)/\tau_D}\right]\\
&\quad + e^{-(t-z'/v)/\tau_D}\left[\left.\frac{di(0,t)}{v^* dt}\right|_{t=z'/v^*} + \frac{i(0,z'/v^*)}{v^*\tau_D}\right], \quad t \ge z'/v
\end{aligned} \tag{23}$$

Equations (22) and (23) are equivalent and give the equivalent corona current per unit channel length for the DU model converted to equivalent LS model. The first term on the right-hand side of (23) represents the fictitious corona current, while the second term represents the negative of the actual corona current for the DU model. Note that the equivalent corona current given by (23) should be negated to make it directly comparable to the actual corona current, which is given by (2).

Substituting (22) and (21) in (20), we get

$$\begin{aligned}
\rho(z',t) &= \frac{i(0,t+z'/c) - i(0,z'/v^*)e^{-(t-z'/v)/\tau_D}}{v}\\
&\quad + \int_{z'/v}^{t}\left[-\left(\frac{1}{v}+\frac{1}{c}\right)\frac{\partial i(0,\tau+z'/c)}{\partial \tau}\right.\\
&\quad \left. + e^{-(\tau-z'/v)/\tau_D}\left.\frac{di(0,t)}{v^* dt}\right|_{t=z'/v^*}\right] d\tau\\
&= \frac{i(0,t+z'/c)}{v} - \frac{i(0,z'/v^*)e^{-(t-z'/v)/\tau_D}}{v}\\
&\quad + \left[-\left(\frac{1}{v}+\frac{1}{c}\right) i(0,\tau+z'/c)\right]_{z'/v}^{t}\\
&\quad + \left[-e^{-(\tau-z'/v)/\tau_D}\frac{\tau_D}{v^*}\left.\frac{di(0,t)}{dt}\right|_{t=z'/v^*}\right]_{z'/v}^{t}\\
&= -\frac{i(0,t+z'/c)}{c} - \frac{i(0,z'/v^*)e^{-(t-z'/v)/\tau_D}}{v}\\
&\quad + \frac{i(0,z'/v^*)}{v^*} + \frac{\tau_D}{v^*}\left.\frac{di(0,t)}{dt}\right|_{t=z'/v^*}\\
&\quad - e^{-(t-z'/v)/\tau_D}\frac{\tau_D}{v^*}\left.\frac{di(0,t)}{dt}\right|_{t=z'/v^*}
\end{aligned}$$

$$= -\frac{i(0,t+z'/c)}{c} - \left[\frac{i(0,z'/v^*)}{v} + \frac{\frac{\tau_D}{v^*} di(0,t)}{dt}\bigg|_{t=z'/v^*}\right] e^{-(t-z'/v)/\tau_D} + \frac{1}{v^*}\left[i(0,z'/v^*)+\tau_D \frac{di(0,t)}{dt}\bigg|_{t=z'/v^*}\right], \quad t \geq z'/v. \tag{24}$$

The charge density given by (24) is identical to the total charge density derived for the DU model by Thottappillil *et al.* [12], who used a general continuity equation as applied to lightning return strokes.

B. TCS Model

The longitudinal current at a given height z' above ground and at a given time t for the TCS model is given in [8] as

$$i(z',t) = i(0,t+z'/c)\, u(t-z'/v) \tag{25}$$

where $i(0,t)$ is the channel-base current. This equation can be obtained from (21) by setting $\tau_D = 0$. Similarly, the equivalent corona current for the TCS model follows directly from (22) when $\tau_D = 0$:

$$i'_{\text{cor}} = -\frac{1}{v^*}\frac{\partial i(0,t+z'/c)}{\partial t}, \quad t \geq z'/v. \tag{26}$$

This equivalent corona current per unit length implies a LS at the channel base.

Substituting (25) and (26) in (20), we get

$$\rho(z',t) = \frac{i(0,t+z'/c)}{v} - \frac{1}{v^*}\int_{z'/v}^{t} \frac{\partial i(0,\tau+z'/c)}{\partial \tau}\, d\tau = -\frac{i(0,t+z'/c)}{c} + \frac{i(0,z'/v^*)}{v^*}, \quad t \geq z'/v \tag{27}$$

which also directly follows from (24) when τ_D is replaced with zero.

C. BG Model

The longitudinal-current equation for the BG model [6] is

$$i(z',t) = i(0,t)\, u(t-z'/v) \tag{28}$$

which can be obtained from (25) by replacing c with ∞. Therefore, the equivalent corona current for the BG model can be obtained from the equivalent corona current derived for the TCS model [see (26)] by replacing c with ∞, that is

$$i'_{\text{cor}} = -\frac{1}{v}\frac{\partial i(0,t)}{\partial t}, \quad t \geq z'/v. \tag{29}$$

Substituting (28) and (29) in (20), we get

$$\rho(z',t) = \frac{i(0,z'/v)}{v}, \quad t \geq z'/v \tag{30}$$

which also directly follows from (27) when c is replaced with ∞.

Equations (24), (27), and (30) are based on (20), which is one of the forms of the continuity equation for LS models. Thus, the TCS, BG, and DU models can be converted to equivalent LS models with equivalent corona currents given by (23), (26), and (29), respectively. Note that equivalent corona currents in the TCS, BG, and DU models expressed in terms of a LS at ground level are different from actual corona currents for these models, as discussed in Section V. On the other hand, both the longitudinal-current distribution along the channel and the total charge-density distribution are each the same for LS and DS formulations.

V. Interpretation of the Corona Current in LS and DS Models

As noted in Section I, the conversion of LS to DS or DS to LS models alters the actual (related to the deposited charge density component) corona current of the model. We will refer to this altered corona current as the equivalent corona current. Each type (LS or DS) of model has its corresponding form of the continuity equation: (1) for DS models and (17) for LS models. The use of a "noncorresponding" continuity equation (1) for LS models or (17) for DS models gives rise to the equivalent corona current. For LS models converted to DS models, the equivalent corona current is the sum of the negated actual corona current and a fictitious corona current. For DS models converted to LS models, this is true only for the DU model. For the TCS and BG models, equations for the equivalent corona current apparently do not involve the actual corona current.

In order to obtain the equivalent corona current per unit channel length for LS models converted to DS models, we substitute (8) into (1) to get

$$i'_{\text{cor}}(z',t) = P(z')\frac{1}{v^*}\frac{\partial i(0,t-z'/v)}{\partial t} - \frac{dP(z')}{dz'} i(0,t-z'/v), \quad t \geq z'/v \tag{31}$$

On the other hand, the actual corona current for LS models is given by (18). It appears from comparison of (18) and (31) that the actual corona current is equal to the second term of (31), with the first term of (31) being a fictitious corona current component. However, note that, according to Fig. 2, different sign conventions for the corona current are used in LS and DS models. For the LS models, corona current flowing out of the channel core is defined as positive, while for the DS models, positive corona current flows into the core. Therefore, the second term of (31) and the right-hand side of (18) have equal magnitudes but opposite signs. It appears that in converting the LS models to DS models one needs to cancel the actual corona current and introduce a new, fictitious one. Although the directions of propagation of the actual corona current in the LS and DS models are opposite, charge of the same sign is effectively transported into the channel core in both types of models. Applying (31) to the TL, MTLL, and MTLE models one can derive equivalent corona currents given by (5a)–(5c).

Similarly, one can derive equivalent corona currents for DS models converted to equivalent LS models. We have done so in Section IV for the DU, TCS, and BG models. The resultant equivalent corona currents given by (23), (26), and (29)

TABLE I
ACTUAL AND EQUIVALENT CORONA CURRENTS PER UNIT CHANNEL LENGTH, $i'_{\mathrm{cor}}(z',t)$, FOR TWO FORMULATIONS OF LS MODELS ($t \geq z'/v$).

Model type/ ID	**LS**	**TL**	**MTLL**	**MTLE**
Longitudinal Current (LS and DS model formulations)	$P(z')i(0,t-z'/v)$	$i(0,t-z'/v)$	$\left(1-\frac{z'}{H}\right)i(0,t-z'/v)$	$\exp(-z'/\lambda)i(0,t-z'/v)$
Actual corona current (LS model formulation)	$-\frac{dP(z')}{dz'}i(0,t-z'/v)$	0	$\frac{i(0,t-z'/v)}{H}$	$\frac{\exp(-z'/\lambda)}{\lambda}i(0,t-z'/v)$
Equivalent corona current (DS model formulation)	$\frac{1}{v^*}\frac{\partial P(z')i(0,t-z'/v)}{\partial t} - \frac{dP(z')}{dz'}i(0,t-z'/v)$	$\frac{1}{v^*}\frac{\partial i(0,t-z'/v)}{\partial t}$	$\left(1-\frac{z'}{H}\right)\frac{1}{v^*}\frac{\partial i(0,t-z'/v)}{\partial t} + \frac{i(0,t-z'/v)}{H}$	$\frac{\exp(-z'/\lambda)}{v^*}\frac{\partial i(0,t-z'/v)}{\partial t} + \frac{\exp(-z'/\lambda)}{\lambda}i(0,t-z'/v)$
Total line charge density (LS and DS model formulations)	$\frac{P(z')i(0,t-z'/v)}{v} - \frac{dP(z')}{dz'}Q(z',t)$	$\frac{i(0,t-z'/v)}{v}$	$\left(1-\frac{z'}{H}\right)\frac{i(0,t-z'/v)}{v} + \frac{1}{H}Q(z',t)$	$\frac{i(0,t-z'/v)}{v}\exp(-z'/\lambda) + \frac{\exp(-z'/\lambda)}{\lambda}Q(z',t)$

*The equivalent corona current should be negated to make it directly comparable to the actual corona current. Also given is total line charge density along the lightning channel, which is the same for both model formulations. Note that $Q(z',t) = \int_{z'/v}^{t} i(0,\tau - z'/v)\,d\tau$ and $v^* = (1/v + 1/c)^{-1}$.

are different from actual corona currents in these models that are given by (2), (3), and by (3) with c replaced by ∞ in the expression for v^*, respectively. Interestingly, for the DU model, the equivalent corona current involves the actual corona current, similar to the LS models converted to DS ones.

Using the concept of equivalent corona current, LS models can be converted to DS models, and DS models can be converted to LS models. This property of engineering return stroke models can be viewed as duality. Both equivalent and actual corona currents for LS and DS models are summarized in Tables I and II, respectively. It is important to note that the equivalent corona current in Tables I and II should be negated in order to make it directly comparable to the actual corona current. Thus, it follows from Table I that the equivalent corona current for LS models converted to DS models is the sum of the negated actual corona current and an additional, fictitious corona current. This is also true for the DU model (see Table II). The equivalent corona current for LS models converted to DS models is based on (1), and that for DS models converted to LS models is based on (17). Interestingly, the magnitude of the fictitious corona current per unit length for either LS or DS models can be expressed as $(\partial i/\partial t)/v^*$, where i is the longitudinal current, and $v^* = (1/v + 1/c)^{-1}$, although for the DS models, the longitudinal-current derivative should be negated.

As noted above, the equivalent corona current is totally or in part fictitious current. For example, there exists only fictitious bipolar corona current in the TL model when it is converted to equivalent DS model. Physically, no radial corona current is expected to exist in the TL model, because the deposited charge-density component for this model is equal to zero. In the MTLL and MTLE models, the equivalent corona current consists of the negated actual corona current and a fictitious corona current. The longitudinal current, equivalent corona current, and individual corona current components for the TL, MTLL, and MTLE models converted to equivalent DS models are illustrated in Fig. 3(a) –(c).

Note that the shape of the actual corona current component is unipolar and the same as that of the longitudinal current i while the fictitious corona current component is a narrow bipolar pulse, which is given, as noted above, by $(\partial i/\partial t)/v^*$. The peak value of the fictitious corona current component in the converted MTLL and MTLE models is much larger than that of the actual corona current component.

Only the actual corona current is related to the deposited charge density component given by (19). This can be shown by taking the time integral of the equivalent corona current given by (31) from 0 to t and comparing the result with (19). Indeed

$$\int_{z'/v}^{t} i'_{\mathrm{cor}}(z',\tau)\,d\tau = P(z')\left(\frac{1}{v}+\frac{1}{c}\right)\int_{z'/v}^{t}\frac{\partial i(0,t-z'/v)}{\partial \tau}\,d\tau - \frac{dP(z')}{dz'}\int_{z'/v}^{t} i(0,\tau-z'/v)\,d\tau \quad (32)$$

that is

$$\int_{z'/v}^{t} i'_{\mathrm{cor}}(z',\tau)\,d\tau = P(z')\left(\frac{1}{v}+\frac{1}{c}\right)[i(0,t-z'/v)-i(0,0)] - \frac{dP(z')}{dz'}\int_{z'/v}^{t} i(0,\tau-z'/v)\,d\tau. \quad (33)$$

TABLE II
ACTUAL AND EQUIVALENT CORONA CURRENTS PER UNIT CHANNEL LENGTH, $i'_{\rm cor}(z', t)$, FOR TWO FORMULATIONS OF DS MODELS

Model type/ ID	**DS/DU**	**TCS**	**BG**
Longitudinal current (DS and LS model formulations)	$i(0, t+z'/c) - i(0, z'/v^*)\exp\left[-(t-z'/v)/\tau_D\right]$	$i(0, t+z'/c)$	$i(0,t)$
Actual corona current (DS model formulation)	$\exp\left[-(t-z'/v)/\tau_D\right]\left[\left.\frac{di(0,t)}{v^* dt}\right\|_{t=z'/v^*} + \frac{i(0,z'/v^*)}{v^*\tau_D}\right]$	$\frac{1}{v^*} i\left(0, \frac{z'}{v^*}\right)\delta(t - z'/v)$	$\frac{1}{v} i\left(0, \frac{z'}{v}\right)\delta(t - z'/v)$
Equivalent corona current (LS model formulation)	$-\frac{1}{v^*}\frac{\partial}{\partial t}\{i(0,t+z'/c) - i(0,z'/v^*)\exp\left[-(t-z'/v)/\tau_D\right]\} + \exp\left[-(t-z'/v)/\tau_D\right]\left[\left.\frac{di(0,t)}{v^* dt}\right\|_{t=z'/v^*} + \frac{i(0,z'/v^*)}{v^*\tau_D}\right]$	$-\frac{1}{v^*}\frac{\partial i(0,t+z'/c)}{\partial t}$	$-\frac{1}{v}\frac{\partial i(0,t)}{\partial t}$
Total line charge density (DS and LS model formulations)	$-\frac{i(0,t+z'/c)}{c} - \left[\frac{i(0,z'/v^*)}{v} + \frac{\tau_D}{v^*}\left.\frac{di(0,t)}{dt}\right\|_{t=z'/v^*}\right]\exp\left[-(t-z'/v)/\tau_D\right] + \frac{1}{v^*}\left[i(0,z'/v^*) + \tau_D\left.\frac{di(0,t)}{dt}\right\|_{t=z'/v^*}\right]$	$-\frac{i(0,t+z'/c)}{c} + \frac{i(0,z'/v^*)}{v^*}$	$\frac{i(0,z'/v)}{v}$

*The equivalent corona current should be negated to make it directly comparable to the actual corona current. Also given is total line charge density along the lightning channel, which is the same for both model formulations. Note that $Q(z', t) = \int_{z'/v}^{t} i(0, \tau - z'/v)\, d\tau$ and $v^* = (1/v + 1/c)^{-1}$.

Assuming that $i(0,0) = 0$ yields

$$\int_{z'/v}^{t} i'_{\rm cor}(z', \tau)\, d\tau = \frac{P(z') i(0, t - z'/v)}{v} + \frac{P(z') i(0, t - z'/v)}{c} - \frac{dP(z')}{dz'}\int_{z'/v}^{t} i(0, \tau - z'/v)\, d\tau. \quad (34)$$

Clearly, the first two terms (which are due to the fictitious corona current component) on the right-hand side of (34) do not exist in (19), with the third term being the deposited charge-density component introduced in [12]. In other words, the first two terms of (34) are not related to the charge deposited in the corona sheath; only the third term is. The fictitious corona current appears to be related to the difference between the transferred (as opposed to deposited) charge densities at the two ends of the differential channel segment, regardless of physical reasons for this difference. In the TL model, this difference has nothing to do with the physical corona-current flow from (or into) the channel core.

Similarly, equivalent corona currents for DS models converted to LS models are different from the actual corona currents derived for the DS models from (1). In the DU model, the equivalent corona current consists of the negated actual and fictitious corona current components, while for the TCS and BG models, equations for the equivalent corona current apparently do not involve the actual corona current. The longitudinal current and equivalent corona current for the DU, TCS, and BG models (also the individual corona current components for the DU model) are illustrated in Fig. 4(a)–(c). As opposed to LS models converted to DS models, for the DU model converted to the equivalent LS model, the two corona-current components initially have opposite direction and the same magnitudes at time $t = z'/v$. Note that actual corona currents in the TCS and BG models involve delta functions, whereas equivalent corona currents (after conversion of these models to equivalent LS models) do not.

Equivalent corona currents defined above, although different from actual corona currents can be used for formal conversion between LS and DS models, since distributions of the longitudinal current and the total charge density along the channel before and after conversion remain the same. This can be viewed as a manifestation of duality of engineering lightning return-stroke models. Conversion of LS models to DS models is particularly useful in extending the model to include a tall strike object, as done in [5].

VI. SUMMARY

We show that any engineering return-stroke model can be expressed using an appropriate continuity equation in terms of either lumped or distributed current sources, with the resultant

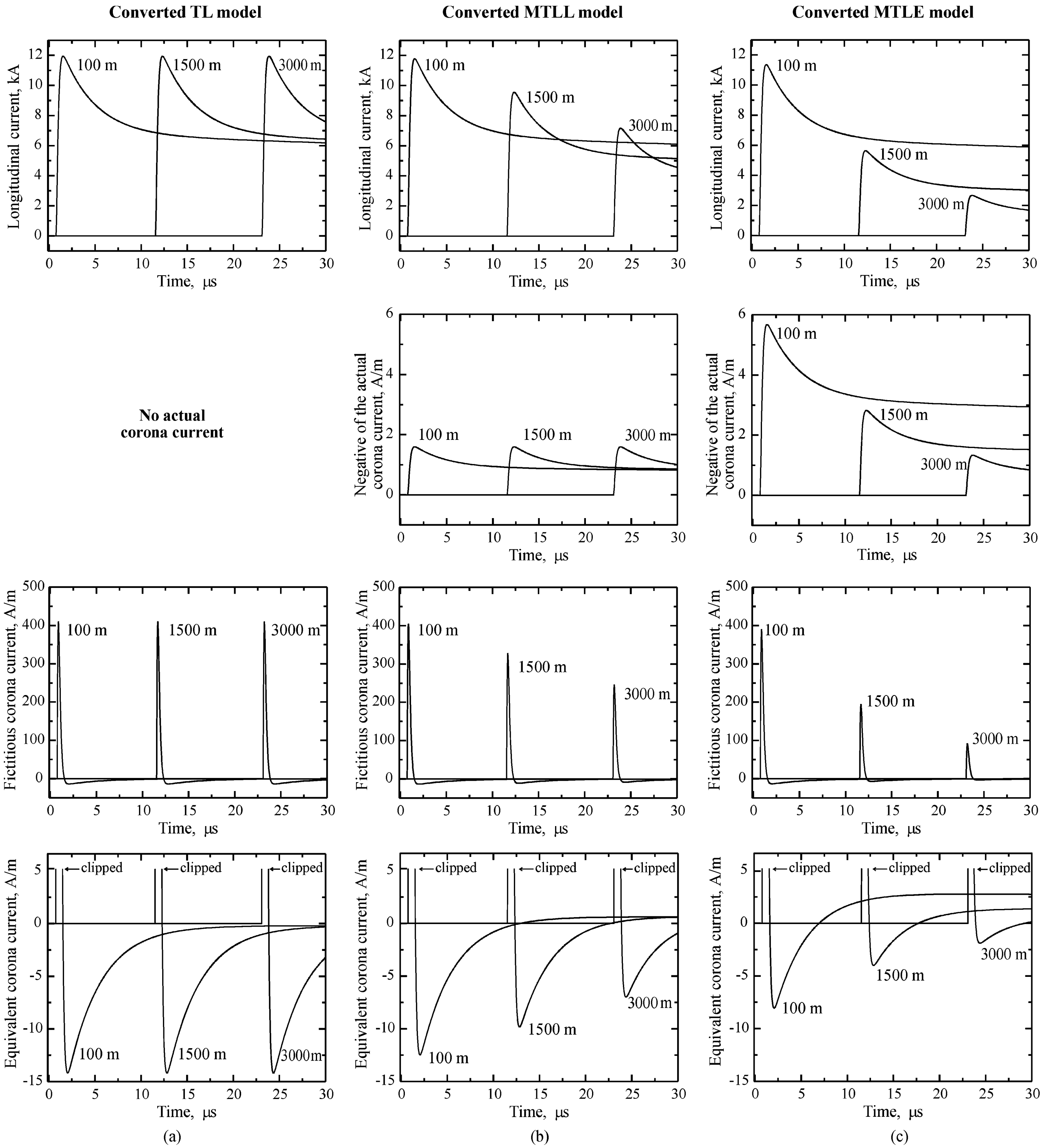

Fig. 3. Longitudinal current (*first row*) and negated actual (*second row*), fictitious (*third row*), and equivalent (*fourth row*) corona currents per unit-length as a function of time at three different heights—100, 1500, and 3000 m—for (a) TL, (b) MTLL, and (c) MTLE models ($H = 7500$ m, $\lambda = 2000$ m, $v = 130$ m/μs.) converted to equivalent DS models. Positive (into the core) equivalent corona current peaks are clipped to accentuate negative (out of the core) overshoots. For the converted TL model, the equivalent corona current is the same as the fictitious corona current but shown with a different vertical resolution. Channel base current used here is the same as that adapted in [14] and is characterized by a current peak of 12 kA and a maximum current rate of rise of about 40 kA/μs.

longitudinal current and total charge density distributions along the channel being the same. This property can be viewed as the duality of engineering models. Conversion of LS models to equivalent distributed source models was previously demonstrated by Rachidi and Nucci [2], Cooray [1], and Rachidi *et al.* [5]. The conversion alters the actual corona current (if any) of the model. For LS models, the actual corona current is unipolar and directed radially out of the channel core, while for DS models, it is unipolar and directed into the channel core. For LS models converted to DS models and for the DU model

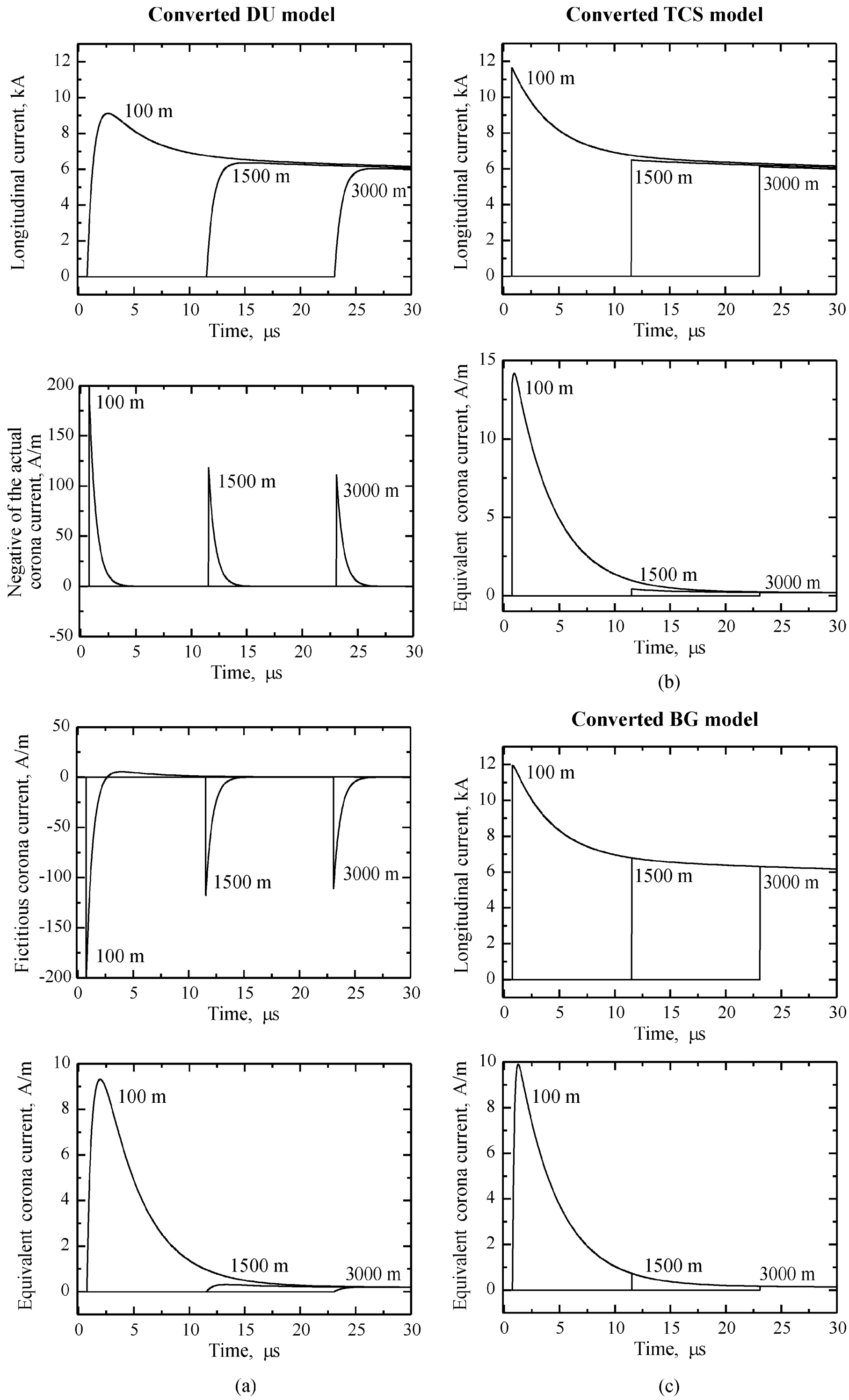

Fig. 4. (a) Longitudinal current, negated actual, fictitious, and equivalent corona currents per unit channel length as a function of time at three different heights—100, 1500, and 3000 m— for the DU model converted to equivalent LS model ($v = 130\ \mathrm{m}/\mu\mathrm{s}, \tau_D = 0.6\ \mu\mathrm{s}$). (b) and (c) Longitudinal current and equivalent corona current per unit channel length for the TCS and BG models converted to equivalent LS models, respectively. Equivalent corona current in the converted TCS and BG models are equal to the corresponding fictitious corona currents. The actual corona currents in the TCS and BG models involve delta functions and are not shown here. Positive corona current flows out of the channel core. Channel base current used here is the same as that adopted in [14].

converted to the equivalent LS model, the corona current is the sum of the negated actual corona current and a fictitious corona current, the latter being bipolar. It appears that in converting these models, one needs to cancel the actual corona current (related to the deposited charge density component) and introduce a new, fictitious one. For the TL model (no longitudinal-current attenuation with height, and hence, no deposited charge-density component) expressed in terms of DSs, there is only a fictitious bipolar corona current component. The fictitious corona current appears to be related to the difference between the transferred (as opposed to deposited) charge densities at the two ends of the differential channel segment, regardless of physical reasons for this difference. In the TL model, this difference has nothing to do with the physical corona current flow from (or into) the channel core. Conversion of the TCS and BG models to equivalent LS models involves replacement of the actual, unipolar corona current with a fictitious one, the latter current being bipolar near the channel base and unipolar at higher altitudes.

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REFERENCES

[1] V. Cooray, "On the concepts used in return stroke models applied in engineering practice," *IEEE Trans. Electromagn. Compat.*, vol. 45, no. 1, pp. 101–108, Feb. 2003.

[2] F. Rachidi and C. A. Nucci, "On the Master, Uman, Lin, Standler and the modified transmission line lightning return stroke current models," *J. Geophys. Res.*, vol. 95, pp. 20389–20393, 1990.

[3] V. A. Rakov and M. A. Uman, "Review and evaluation of lightning return stroke models including some aspects of their application," *IEEE Trans. Electromagn. Compat.*, vol. 40, no. 4, pp. 403–426, Nov. 1998.

[4] V. A. Rakov and M. A. Uman, *Lightning: Physics and Effects*. Cambridge, U.K.: Cambridge Univ. Press, 2003.

[5] F. Rachidi, V. A. Rakov, C. A. Nucci, and J. L. Bermudez, "Effect of vertically extended strike object on the distribution of current along the lightning channel," *J. Geophys. Res.*, vol. 107, pp. ACL 16-1–16-6, 2002.

[6] C. E. R. Bruce and R. H. Golde, "The lightning discharge," *J. Inst. Electr. Eng.*, vol. 88, pp. 487–520, 1941.

[7] G. Diendorfer and M. A. Uman, "An improved return stroke model with specified channel base current," *J. Geophys. Res.*, vol. 95, pp. 13621–13644, 1990.

[8] F. Heidler Zurich, "Travelling current source model for LEMP calculation," in *Proc. 6th Int. Zurich Symp. EMC*, vol. 29F2, Zurich, Switzerland, 1985, pp. 157–162.

[9] M. A. Uman and D. K. McLain, "Magnetic field of lightning return stroke," *J. Geophys. Res.*, vol. 74, pp. 6899–6910, 1969.

[10] V. A. Rakov and A. A. Dulzon, "Calculated electromagnetic fields of lightning return stroke," *Tekh. Elektrodinam.*, no. 1, pp. 87–89, 1987.

[11] C. A. Nucci, C. Mazzetti, F. Rachidi, and M. Ianoz, "On lightning return stroke models for LEMP calculations," presented at the 19th Int. Conf. Lightning Protection, Graz, Austria, Apr. 1988.

[12] R. Thottappillil, V. A. Rakov, and M. A. Uman, "Distribution of charge along the lightning channel: Relation to remote electric and magnetic fields and to return-stroke models," *J. Geophys. Res.*, vol. 102, pp. 6987–7006, 1997.

[13] V. A. Rakov and A. A. Dulzon, "A modified transmission line model for lightning return stroke field calculation," in *Proc. 9th Int. Zurich Symp. Electromagn. Compat.*, Zurich, Switzerland, Mar. 1991, pp. 229–235.

[14] G. Maslowski and V. A. Rakov, "A study of the lightning channel corona sheath," *J. Geophys. Res.*, vol. 111, p. D14110, 2006.

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