On the electric field at the tip of dart leaders in lightning flashes

Vernon Cooray, Marley Becerra, Vladimir Rakov

1 Division for Electricity and Lightning Research, Ångström Laboratory, University of Uppsala, Box 534, SE 751 21, Uppsala, Sweden
2 Department of Electrical and Computer Engineering, University of Florida, Gainesville, USA

1. Introduction

Lightning dart leaders can be treated as self propagating discharges carrying a high electric field at their tip. These intense fields at the tip lead to enhanced collisional ionization and to photo-ionization at the tip of the discharge. Knowledge on the magnitude and the spatial extent of the electric field facilitating the continuous propagation of the dart leader may help in understanding the underlying physical mechanism of dart leaders and in providing explanations for their experimentally observed features.

Cooray (1996) examined the correlation between the electric field at the tip of the dart leader and its speed of propagation. Based on a dart leader radius of 1 cm, he estimated that the electric field at the leader tip should be about a factor 1.5 larger than the breakdown electric field of the low-density air in the defunct return stroke channel along which the dart leader is propagating. Further, Jordan et al. (1997) estimated that the peak power dissipated by a typical dart leader is about 300–500 MW/m and the energy dissipated within the first 10 ms or so is about 500–600 J/m. Furthermore, the minimum resistance and the maximum radius of the core of a typical dart leader are estimated to be about 3 Ω/m and 0.003 m, respectively.

2. Mathematics of the problem

The geometry of the problem under consideration is shown in Fig. 1. Let us assume that at time \( t = t_0 \) the dart leader tip is located at a height \( z_0 \) above ground level. We would like to evaluate the longitudinal electric field at points on the axis of the dart leader. Let us consider the point \( P \) located on the axis of the dart leader. The axial electric field produced at this point by the dart leader element \( dz \) is given by Cooray et al. (1989)

\[
E(t_o) = E_I(t_o) + E_E(t_o) + E_\phi(t_o)
\]

\[
E_I(t_o) = -\frac{1}{2\xi_0} \left[ \int_0^{R_{it}t_o} \cos^2 \theta \frac{2}{t_f} \int_{t_o}^{t_z-t/o} f\left(z, \tau - t_o\right) d\tau \right]
\]

\[
+ \left( \int_0^{R_{it}t_o} \cos^2 \theta \frac{2}{ct} \int_{t_o}^{t_z-t/c} f\left(z, \tau - t_o\right) d\tau \right) dx \right] dz 
\]

\[
-\frac{1}{2\xi_0} \left[ \int_0^{R_{it}t_o} \sin^2 \theta \frac{1}{t_f} \int_{t_o}^{t_z-t/o} f\left(z, \tau - t_o\right) d\tau \right]
\]

\[
+ \left( \int_0^{R_{it}t_o} \sin^2 \theta \frac{1}{ct} \int_{t_o}^{t_z-t/c} f\left(z, \tau - t_o\right) d\tau \right) dx \right] dz 
\]

\[
-\frac{1}{2\xi_0} \left[ \int_0^{R_{it}t_o} \sin^2 \theta \frac{1}{ct^2} \frac{d}{dt} \left( f\left(z, t - t_o - \frac{t}{c}\right) \right) \right] dx \right] dz
\]
where $E_r(t_0)$ is the electric field at point $P$ generated by the channel element, $E_s(t_0)$ the electric field generated by its image accounting for the perfectly conducting ground plane, $E_d(t_0)$ the background electric field produced by the charges in the thundercloud at the point of interest, $v$ the speed of the dart leader, $H$ the height at which the dart leader is originated, $I$ the current in the dart leader channel which is assumed to be distributed uniformly over the core of the dart leader channel, $J$ the dart leader current density, $c$ the speed of light, $v$, the speed of streamers forming the corona sheath, $R$, the radius of the cora of the dart leader, $R_c$ the radius of the corona sheath at a given time, $R_{cor}$ the maximum radius of the corona sheath, $\rho(z)$ the charge per unit length on the dart leader at height $z$, and $E_c$ the breakdown electric field in air. Other geometrical parameters $x$, $r$, $\theta$, and $\zeta$ are shown in Fig. 1. In these equations, it is assumed that the radius of the corona sheath expands to its maximum value in a time approximately equal to the time taken by the streamers to travel the full width of the corona sheath. Note that, since the dart leader propagates at a speed close to $10^7$ m/s, the streamers propagating at a speed of about $2 \times 10^4$ m/s are not capable of generating a corona region ahead of the dart leader tip. Moreover, in writing down Eqs. (2) and (3) it is assumed that the charge on the channel is located in the corona sheath whereas the longitudinal current is confined to the core of the leader channel. In the calculations to be followed it is assumed that $v_c = 2 \times 10^5$ m/s and $E_c = 3.0 \times 10^6$ V/m (Cooray, 2003). The electric field at any point on the axis of the dart leader channel can be obtained by summing the contributions from all the elements of the channel. Our calculations show that for dart leader tip heights larger than a few meters the contribution to the electric field at any point on the axis of the dart leader channel is negligible. The solution of Eqs. (1) and (2) requires an expression for the dart leader current as a function of time and height. We will treat this problem in the next section.

### 3. Current of the dart leader

To evaluate the current in the dart leader, we will adopt a concept similar to that used to describe the temporal variation of the return stroke current in engineering return stroke models belonging to current generation type (Cooray, 2003), also referred to as the traveling current source type (Rakov and Uman, 2003). The current generation model return stroke models assume that the return stroke current is generated by the neutralization of the...
charge deposited by the leader in the corona sheath. In these models, it is hypothesized that the neutralization process gives rise to an elementary current at each channel section. These elementary currents are assumed to travel to ground with the speed of light. The total return stroke current is the cumulative effect of these elementary current sources. However, in contrast to the return stroke models which assume that the elementary current sources are driven by the discharge of the corona sheath, the elementary current sources of the dart leader are driven by the charging of the deflect return stroke channel during the passage of the dart leader. As the dart leader passes through a given section of the channel, the change in the potential requires a certain amount of negative charge to be deposited on this channel section. As a result, a wave of positive current (or electron deficiency) travels towards the cloud leaving a net negative charge on the newly created channel section. Since this current transports positive charge into the cloud, each newly created dart leader channel section acts as a current source which effectively drains negative charge from the charge center in the cloud. A similar scenario has been considered by Bazelyan (1995), Cooray (1996) and Thottappillil et al. (1997) in modeling the dart leader current.

We assume that the current source at a given point on the channel is turned on by the passage of the dart leader tip through that point and once turned on each of these current sources generates a current pulse that travels to cloud with the speed of light. For the sake of simplicity, we also assume that the amplitude of the corona current pulse associated with a given elementary channel section decays exponentially with time with a decay time constant $\tau$. Then, the corona current per unit length generated by a channel element located at height $z$ (see Fig. 1) can be written as

$$I_\text{d}(z, t) = I_\text{d}(z) e^{-(t-(H-z))/\tau} \quad t > (H-z)/v$$

where $v$ is the speed of the dart leader and $H$ is the height of origin of the dart leader. Since the integral of the corona current per unit length at a given height from $t = 0$ to $t = \infty$ is equal to the charge per unit length of the leader at that height, $\rho(z)$ and $I_\text{d}(z)$ are related to each other through the equation

$$I_\text{d}(z) = \rho(z)/\tau(z)$$

Recently, Cooray et al. (2007) conducted a study to evaluate the charge distribution along the dart leader channel as a function of the prospective subsequent return stroke current. They have fitted the following analytical expression for the numerically calculated charge distribution

$$\rho(z) = k \left(1 - \frac{\xi}{H-z_0}\right) G(z_0) I_p + \frac{I_p(a + b_1)}{1 + c_1 + d_1} L(z_0)$$

$$\xi = z - z_0$$

$$G(z_0) = 1 - (z_0/H)$$

$$L(z_0) = 0.3\xi + 0.7\beta$$

$$\alpha = e^{-\xi z_0 - 0.5\beta}$$

$$\beta = \left(1 - \frac{z_0}{H}\right)^{1/3}\frac{\beta_1}{\beta_2}$$

where $z_0$ is the height of the leader tip above ground in meters ($z_0 > 10$ m), $H$ the height of the cloud charge source in meters, $\rho(z)$ the charge per unit length (in C/m), $\xi$ (in meters) the length along the dart leader channel with $\xi = 0$ at the tip of the leader, $I_p$ the subsequent return stroke peak current in Amperes associated with the dart leader, $k = 5.09 \times 10^{-8}$ s/m, $a = 1.325 \times 10^{-8}$ s/m, $b = 7.06 \times 10^{-8}$ s/m$^2$, $c = 2.089$ m$^{-1}$, and $d = 1.492 \times 10^{-2}$ m$^{-2}$, $\beta_1 = 10.0$ m and $\beta_2 = 75.0$ m. The functions $G(z_0)$ and $L(z_0)$ are dimensionless. It is important to mention here that in deriving the dart leader charge distribution, Cooray et al. (2007) have assumed that the electric field between the cloud charge region and the ground is uniform. This assumption is valid if the cloud charge region has a very large horizontal extent, compared to its height above ground. They have provided references that validate this assumption to some extend. The important point is that the charge distribution on the leader channel, especially on the sections of the channel close to ground, is determined mainly by the potential of the leader channel (i.e., cloud potential) and not by the particular geometry assigned to the charge distribution in the cloud.

Eqs. (11)–(18) determine completely the current along the dart leader. However, one needs to know the value of $\tau$ to evaluate the dart leader current. Calculations show that the risetime of the dart leader current is controlled by the duration of the corona current; the longer the duration of the corona current, the larger the risetime of the dart leader current. Since the duration of the corona current is determined by $\tau$, one can expect a relationship between this and the risetime of the dart leader current. Indeed, our calculations show that the 20–80% risetime of the dart leader current is approximately equal to $\tau$. Now, the estimates of dart leader current risetimes obtained by Cooray et al. (1989) and Jordan et al. (1997) show that the 20–80% risetime of the dart leader current at channel sections close to ground is about 0.5–1 $\mu$s. Cooray et al. (1989) also found that the risetime of the dart leader current did not change significantly over the heights up to about 800 m above ground. This sets the value of $\tau$ to about 0.5–1 $\mu$s.

The dart leader current distribution along its channel when the dart leader tip is located at 250 m above ground level is shown in Fig. 2 for three values of $\tau$ (in the calculations $I_p = 12$ kA). Note that a dart leader that results in a 12 kA return stroke is characterized by a peak current of about 1 kA in the vicinity of the ground. This is in agreement with the inference made by Idone and Orville (1985) based on the correlations observed between the optical intensities of dart leaders and return strokes. Somewhat larger dart leader currents, 2–4 kA, were reported by Kodali et al. (2005).
reaches a peak value within a fraction of a microsecond and then decays to a more or less steady value within a few microseconds. Now, to obtain the total field one has to add the background electric field which is unknown. Unfortunately, the background field may vary from one flash to another and, to make the matters worse, what we are interested here is the background electric field present during the propagation of dart leaders. In this paper, we utilize the following approach to overcome this situation. The experimental data obtained from long sparks with currents having 0.1 μs rise time and 100 μs duration show that the potential gradient of spark channels also behaves in a manner similar to the one shown in Fig. 3a (Montano et al., 2006). The experimental data show that the steady value of the electric field remains more or less constant and close to 2500 V/m as the peak amplitude of the current varies between 0.5 and 2.5 kA. Based on this result, the value of the background electric field is selected so that the amplitude of the steady value of the dart leader electric field reaches 2500 V/m. The total electric field obtained in this manner is depicted in Fig. 3a by a dotted line. As noted earlier, this change does not change the maximum value of the electric field.

Fig. 4 depicts the electric field at a point on the axis of the dart leader channel located at a height of 250 m above the ground level as the dart leader passes through that point. The calculations are presented for three values of corona decay time constant, τ. In these calculations, we have assumed that the speed of the dart leader is 107 m/s which is the typical dart leader speeds observed in experimental studies (Jordan et al., 1992). On the horizontal axis of this figure, we have the distance to the dart leader tip from the point of observation. This distance is considered to be negative when the dart leader tip is above the point of observation and positive when it is below. The variation of the field as a function of time is shown in Fig. 5. Recall again that zero time corresponds to the instant when the dart leader tip coincides with the point of observation. Note that the electric field rises very sharply as the dart leader passes through the observation point, reaches the peak soon after that and then decays to a relative low level. Observe that the rapid increase in the electric field takes place within the first few meters behind the dart leader front. For the three rise times of the dart leader current considered in these calculations (i.e., for different values of τ), the peak electric field associated

with the dart leader varies between (0.7–3.0) × 106 V/m. Miki et al. (2002) measured dart leader electric field peaks ranging from 0.2 × 106 to 1.5 × 106 V/m, with the median being 0.6 × 106 V/m, at distances ranging from 0.1 to 1.6 m from the lightning channel. Observe also that the full spatial width of the electric field pulse increases with increasing risetime of the dart leader current. For the three current rise times considered the spatial width of the electric field pulse (measured by drawing a
The horizontal line at the electric field value corresponding to 4750 m varies from about 10 m to about 25 m. Note that (in Fig. 3b) the full width at half maximum of the electric field is about 130, 200 and 480 ns for dart leader current risetimes of 0.5, 1.0 and 2.0 ms, respectively.

The results presented in Figs. 4 and 5 are for a dart leader that results in a typical subsequent return stroke (i.e., a stroke with a 12 kA peak current). The electric field corresponding to dart leaders giving rise to other return stroke peak currents would scale linearly with the peak current. Fig. 6 depicts the peak electric field of the dart leader as a function of its peak dart leader current. Observe that the electric field in the dart leader channel increases with increasing dart leader current and it scales linearly with the peak current. This is the case since according to the results of Cooray et al. (2007) the linear charge density on the leader channel scales linearly with the prospective subsequent return stroke current. This one can also see from Eq. (13).

Experimental data of triggered lightning also indicate that the charge density on the dart leader channel scales linearly with the return stroke peak current (Rakov, 1999). It is important to note, however, that the relationship between the peak dart leader electric field and the peak dart leader current remains linear as long as we assume that the risetime of the dart leader current and the speed of the dart leader current remains the same with increasing current.

Dwyer et al. (2003) reported measurements of bursts of X-rays, with energies up to about 250 keV, originating from dart leaders in rocket-triggered lightning. In recent years, the relativistic runaway electron breakdown model (Gurevich and Zybin, 2001) has gained popularity in explaining the generation of X-ray bursts from lightning processes. However, the calculations performed by Dwyer (2004) show that this model is inconsistent with the observed spectrum and flux of the X-ray emission from dart leaders. Their results imply that the so-called cold runaway electron breakdown model may explain the experimental observations. In the cold runaway electron breakdown model, the runaway electrons are produced from the bulk of the free electron population. However, the cold runaway electron breakdown model requires the electric field to be much higher than the conventional breakdown electric field in the medium. Moreover, the electric field enhancement should occur very quickly, since otherwise the ionization and charge transport will neutralize the field preventing the cold runaway breakdown from occurring. At normal atmospheric air density, the electric field required to push electrons to the runaway mode is about $2.6 \times 10^7$ V/m. This threshold decreases linearly with decreasing air density. The dart leader is propagation along a defunct return stroke channel where the air temperature should be in the range 3000–5000 K.
(Uman and Voshall, 1968). Since the defunct return stroke channel is at atmospheric pressure, the density of air in the defunct channel decreases with increasing temperature, the threshold electric field necessary for electron runaway decreases with increasing temperature. In air density corresponds to 3000 K at atmospheric pressure, this threshold electric field is about $2.6 \times 10^6 \text{ V/m}$ and at 5000 K it is about $1.5 \times 10^6 \text{ V/m}$. The results presented in Figs. 3–6 show that as the dart leader tip passes a given point on the channel the electric field at that point may increase for a very short time to values higher than the threshold electric field necessary for cold electron runaway in low-density air in the pre-dart leader channel. This finding is in support of the hypothesis of Dwyer (2004) that the cold runaway electron breakdown may be at work in dart leaders. It is worth noting that X-ray burst, similar to those produced by dart leaders, were also observed in association with stepped leaders propagating through virgin air, of natural lightning (Dwyer et al., 2005).

### 4.2. Power dissipation in the dart leader channel

The temporal variation of the dart leader electric field can be combined with the dart leader current to estimate the electric power per unit length generated by the dart leader. The power generated by a typical dart leader of 1 kA peak current with two different rise times is shown in Fig. 7. The results show that the peak power generated by a typical dart leader lies in the range $3 \times 10^8$–$5 \times 10^8 \text{ W/m}$. Also observe that the peak power generated by a dart leader increases with decreasing risetime. It is also important to recall that the calculation is for a typical dart leader current and that the peak power generated will increase linearly with increasing the dart leader peak current.

### 4.3. Energy dissipation in the dart leader

The energy dissipation in the dart leader as a function of time can be obtained by integrating the power curve. It is important to note in this respect that, depending on the location of the point of observation, the time available for the dart leader to dissipate energy is limited to the time interval between the passage of the dart leader through the point of observation and the subsequent arrival of the return stroke front at that point. For example, at a point located at 250 m from ground level this time interval is limited to 25 μs if the downward propagating speed of the dart leader is $10^7 \text{ m/s}$. Figs. 8 depicts the energy dissipation at a point located 250 m from ground level on the path of the dart leader, as the dart leader passes through this point, for two different dart leader current risetimes. The results show that during the first 10 μs, a typical dart leader releases about 500–600 J/m. Note that the full width of the electric field corresponding to a dart leader having a 0.5 μs risetime is about 1 μs (see Fig. 5). During this time, the dart leader dissipates about 350 J/m. This amount of energy can be interpreted as the energy needed to convert the dart leader channel supporting a 1 kA current from essentially non-conducting state to the highly conducting state.

### 4.4. The resistance and the radius of the hot core of the dart leader

Once the electric field in the dart leader channel and the corresponding current are known, one can evaluate how the resistance of the dart leader channel evolves as a function of time. Fig. 9 depicts how the resistance of the dart leader channel decreases with time. Note that the initial resistance of the dart leader channel is very high but it decreases to a more or less steady value within about 10 μs. For a typical dart leader, the minimum resistance obtained is about 3 Ω/m.

Recall that the calculation of the electric field inside the dart leader channel requires the radius of the core of the dart leader channel as an input parameter. In reality, the hot core of the dart leader channel may vary with time. The initial radius may lie in the millimeter range and as the current passes through the channel the radius may expand gradually to a value which is in the centimeter range. Fig. 10 depicts the electric field at a point located at a height of 250 m from ground level located on the path of the dart leader as the latter passes through it. Calculations are presented for two values of dart leader core radius, 0.001 m and 0.02 m. Observe that except for the initial peak, which occurs

![Fig. 7](image-url)

**Fig. 7.** The temporal variation of the power dissipation at a point located 250 m above ground as the dart leader passes through the point. The zero time corresponds to the instant when the dart leader tip coincides with the point of observation. Curves (a) and (b) correspond to dart leader rise times of 0.5 and 1 μs, respectively.

![Fig. 8](image-url)

**Fig. 8.** The temporal variation of the energy dissipation at a point located 250 m above ground as the dart leader passes through the point. The zero time corresponds to the instant when the dart leader tip coincides with the point of observation. Curves a and b correspond to dart leader rise times of 0.5 and 1 μs, respectively.
within a fraction of a microsecond, the two electric fields are almost identical to each other for longer times. This shows that the electric field is not that sensitive to the radius of the core of the dart leader. This fact can be utilized to evaluate how the radius of the hot core of the dart leader varies with time. For example, the radius of the hot core of the dart leader channel is given by

\[ r(t) = \sqrt{\frac{I(t)}{\pi \sigma(t) E(t)}} \]  

(19)

where \( I(t) \) is current flowing through the channel, \( \sigma(t) \) the conductivity and \( E(t) \) the electric field in the channel. Now, the conductivity of air remains more or less constant around \( 10^{-5} \) \( \text{S/m} \) over the temperature range \( 10^4 \text{–} 2 \times 10^4 \text{K} \). Thus, assuming that the temperature of the channel is kept within this range as the channel expands one can replace the time varying conductivity by the above constant value. The variation of channel radius as a function of time calculated using this equation is shown in Fig. 11 for a typical dart leader with 1 kA peak current. Note that the radius increases with increasing dart leader current and for a typical dart leader with 1 kA current the radius after 10 \( \mu \text{s} \) is about 3 mm. Of course, the assumption of constant and large conductivity assumed in the above calculation may not be valid during the rising part of the electric field including the peak. But since the electric field relaxes to a small value within a microsecond or so the high conductivity assumption may be reasonable for times longer than this.

5. Conclusions

The results obtained in this study show that as the dart leader tip passes a given point on the defunct return stroke channel the electric field increases within a fraction of a microsecond to values larger than the critical electric field necessary for the initiation of cold electron runaway in low-density air comprising the channel. These results are in support of the hypothesis that cold runaway electron breakdown may play a role in the emission of X-ray bursts by dart leaders. The calculations also show that a typical dart leader generates a peak power of about \( 5 \times 10^8 \text{W/m} \) and the energy dissipation in the first 10 \( \mu \text{s} \) is about 500 J/m. According to the results, the minimum resistance of the dart leader channel is about 3 \( \Omega/\text{m} \). Using the simplifying assumption that the conductivity of the channel remains more or less constant during the expansion phase, the temporal variation of the radius of the hot core of the dart leader channel is derived. The results indicate that the maximum radius of the hot core of the dart leader is about 3 mm.

Acknowledgements

The research work reported here is funded partly by the Swedish Research Council (Grant no. G-EG/GU 1448-306), the Swedish foundation for International Cooperation in Research and Higher Education (STINT) (Grant no. IG2004-2031), a donation to
References


Cooray, V., 2003. The Lightning Flash: IEE.


Thottappillil, R., Rakov, V.A., Uman, M.A., 1997. Distribution of charge along the lightning channel: Relation to remote electric and magnetic fields and to return stroke models. J. Geophys. Res. 102, 6887–7006.