Expressions for far electric fields produced at an arbitrary altitude by lightning return strokes

Rajeev Thottappillil, Vladimir A. Rakov, and Nelson Theethayi

1. Introduction

There are some applications where knowledge of lightning return stroke fields above ground is needed. For example, modeling of “elves”, one type of mesospheric transient luminous events, requires computation of transient return stroke fields (and Poynting vectors) at high altitudes [e.g., Rowland et al., 1995]. Various return stroke models are used for the calculation of remote fields. A description of commonly used return stroke models can be found in the work of Rakov and Uman [1998]. Some of the models have been validated previously [e.g., Nucci et al., 1990; Thottappillil and Uman, 1993; Thottappillil et al., 1997] using measured remote fields, and, when available, optically measured return stroke speeds and measured currents. Perhaps the most widely used are the transmission line (TL), modified transmission line model with exponential decay (MTLE), and modified transmission line model with linear decay (MTLL) models. Analytical expressions for far fields at an arbitrary elevation angle based on the TL model are well known [e.g., Levine and Willett, 1992; Krider, 1992; Thottappillil et al., 1998; Rakov and Tuni, 2003]. Deriving similar expressions for more realistic MTLL and MTLE models is of interest. Nucci et al. [1990] has presented an analytical expression for far radiation field, but only at ground, for the MTLE model. Recently, Shao et al. [2005] have derived, for the MTLE model, an expression for radiation field at an arbitrary elevation angle without taking into account the presence of ground. Shao and Heavner [2006] used their expression in modeling of bipolar electric field pulses associated with the lightning preliminary breakdown process. Electric field expression for the MTLE model (including the effect of perfectly conducting ground) for the special case of step function current wave was derived by Wait [1998] and employed by Rakov and Tuni [2003].

In this paper, we derive expressions for far (radiation) fields at an arbitrary elevation angle for both the MTLE and MTLL models and test their validity against exact general expressions that include all the terms (static, induction, and radiation). Field expressions are derived for free space; that is, neglecting interaction of electromagnetic waves with conducting atmosphere, which should be important at ionospheric altitudes. Additionally, we demonstrate in this paper that different, containing either spatial or time integral, but equivalent equations can be derived for each of the models. We also show that the expression for the MTLE model derived by Shao et al. [2005] is equivalent to the corresponding expressions derived here. Note that the fields above ground presented by Thottappillil and Rakov [2006] using Shao et al.’s equation are incorrect, because of an error in the computer code.

2. Far Electric Field Equations for Transmission Line Type Models

The relationship between the channel-base current and current at any height along the channel in the trans-
mission line type (TL, MTLE, and MTLL) models is defined as [e.g., Rakov and Uman, 1998]

\[ i(z', t) = i(0, t - z'/v) \cdot P(z') \]  

(1)

where \( P(z') \) is the current attenuation factor, and \( v \) is the upward return stroke speed. In the original TL model, the current is assumed to travel without any attenuation [Uman and McLain, 1969]. Nucci et al. [1988] proposed an exponential function for current decay and Rakov and Dulzon [1987, 1991] proposed a linear function for current decay with height. Relationships between the channel-base current and current at any height \( z' \) for the three return stroke models introduced above are given by

TL \[ i(z', t) = i(0, t - z'/v) \]  

(2)

MTLE \[ i(z', t) = i(0, t - z'/v) \cdot e^{-z'/\lambda} \]  

(3)

MTLL \[ i(z', t) = i(0, t - z'/v) \cdot (1 - \frac{z'}{H}) \]  

(4)

where \( z' \) is the height above ground, \( t \) is the time, \( v \) is the return stroke speed, \( \lambda \) is the current decay height constant, and \( H \) is the total length of the return stroke channel. Although the MTLE and MTLL models are the focus of this paper, we also include equations for the TL model, in order to facilitate direct comparison.

[5] The exact expression for the electric field at any point in space has been presented in a generalized form by Thottappillil et al. [1998]. Far from the channel the field is dominated by the radiation field component, traditionally identified as that containing \( c^{-2} R^{-1} \) and is given by [e.g., Thottappillil et al., 1998]

\[ E_{\text{far}} \approx E_{\text{rad}}(r, \theta, \phi) \]

\[ = \frac{1}{4 \pi \varepsilon_0} \int_0^{L(t)} \alpha(L') \sin \alpha(L') \frac{dL(t) - R(L')/c}{c^2 R(L')} dL' \]

\[ + \frac{1}{4 \pi \varepsilon_0} \int_0^{L(t)} \sin \alpha(L') i(L', t - R(L')/c) dL' \]  

(5a)

where \( L(t) \) is the length of channel contributing to the field at point \( P \) at time \( t \), \( R(z') \) is the inclined distance between the channel segment at height \( z' \) and point \( P \), \( \alpha(z') \) is the angle between the vertical channel and the inclined distance \( R(z') \) (see Figure 1), and \( c \) is the speed of light. The effect of ground plane is not included. The exact expression for \( L(t) \) can be obtained as the solution of \( t = \frac{L'(t)}{v} + \frac{R(L')}{c} \), where \( R(L') = \sqrt{L^2(t) + r^2 - 2L(t)r \cos \theta} \), and is given by Thottappillil et al. [1998, equation (25)]. Angle \( \theta \) is the angle between the vertical channel and the line connecting the channel base and point \( P \) (see Figure 1). An approximate equation for \( L'(t) \) that is valid at far distances; that is, when \( L(t) \ll r \), is given by [e.g., Thottappillil et al., 1998]

\[ L'(t) \approx \frac{v \cdot (t - r/c)}{1 - \frac{r}{c} \cos \theta} \]  

(5b)

The geometrical parameters used in equations (5a) and (5b) are illustrated in Figure 1. The second term in (5a) is nonzero only if there is a current discontinuity at the return stroke front. The total electric field is given by (B1) in Appendix B, from which equation (5a) can be obtained by retaining only radiation field terms (fifth and sixth terms of (B1)), which dominate at far distances and at early times. It is worth noting that for small values of \( \theta \) and at intermediate distance the induction field component gives the primary contribution to the total field. Lu [2006], who computed electric fields due to return strokes at a height of 90 km, found that the induction field component dominates within a radius of about 11 km (depending on the return stroke speed) of the vertical lightning channel. Also, relative magnitude of the electrostatic component increases at later times. Note also that the traditional division of the total field into static, induction (or intermediate), and radiation field components, on the basis of distance dependence, is not unique, especially at close and intermediate distances, as demonstrated by Thottappillil and Rakov [2001a, 2001b].

[6] Equations (2), (3), and (4) can be substituted in (5a) and the latter can be simplified to get approximate expressions for far radiation field corresponding to the TL, MTLE, and MTLL model, respectively. Details of the steps leading to the simplified expressions are given in Appendix A. The final approximate expressions, accounting for current discontinuity at the front (if any), are,

\[ E_{\text{farTL}} = \hat{\theta} \frac{1}{4 \pi \varepsilon_0 c^2 r} \frac{v \sin \theta}{1 - \frac{v}{c} \cos \theta} \]

\[ \cdot \left[ i(0, t - \frac{r}{c}) - \frac{1}{\lambda} \int_0^{L(t)} i(L', t - R(L')/c) \cdot e^{-z'/\lambda} dz' \right] \]  

(7a)

\[ E_{\text{farMTLE}} = \hat{\theta} \frac{1}{4 \pi \varepsilon_0 c^2 r} \frac{v \sin \theta}{1 - \frac{v}{c} \cos \theta} \]

\[ \cdot \left[ i(0, t - \frac{r}{c}) - \frac{1}{H} \int_0^{L(t)} i(L', t - R(L')/c) dz' \right] \]  

(8a)
by the corresponding image expression (not given in this paper). The channel-base current shown in Nucci et al. [1990, Figure 4], which is representative of subsequent return strokes was used in the calculations. Results are presented for the return stroke speed $1.5 \times 10^8$ m/s in Figure 2 for the MTLE model and in Figure 3 for the MTLL model. In the calculations, the current decay height constant $\lambda$ in the MTLE was assumed to be 2000 m, and the constant $H$ in MTLE model was 7500 m. One can see that predictions of approximate expressions (7a) and (7b) and (8a) and (8b) for far electric fields (dotted line) are in very good agreement with those of the exact expression (solid line). In fact, the curves based on approximate expressions are almost indistinguishable from exact ones, which indicate that (1) equations (7a) and (7b) and (8a) and (8b) are correctly derived from (B1) (and its image counterpart) and (2) fields at 100 km and $\theta$ ranging from 10° to 90° are indeed dominated by the radiation field component. In this paper whenever the effect of ground is considered, it is assumed to be perfectly conducting. Real ground has finite conductivity, which gives rise to field propagation effects [e.g., Cooray, 2003; Shoory et al., 2005]. These effects may be significant at ground surface ($\theta = 90^\circ$), but are expected to be small at high altitudes considered here.

[8] Equations (7a) and (7b) and (8a) and (8b), corresponding to the MTLE and MTLL models, respectively, are new. When $\lambda = \infty$ in (7a), it reduces to (6a) corresponding to the TL model. Similarly, when $H = \infty$ in (8a), it reduces to (6a). Rachidi and Nucci [1990] had derived an expression relating the channel-base current and the far (radiation) fields for the MTLE model (including the effect of perfectly conducting ground) for the special case of field point at ground ($\theta = 90^\circ$) and assuming no current discontinuity at time $t = 0$. Their expression is equivalent to the sum of (7a) and (7b), if $\theta = 90^\circ$.

[9] We now consider the special case of a step function current wave, $I_0 e^{-t/\lambda}$, $u(t - z'/v)$, that travels upward at the corresponding image expression.
speed \( v \) and whose magnitude decreases with height as \( e^{-z/\lambda} \) (the MTLE model). The radiation field (\( \theta \) component) equation for this special case (including the effect of perfectly conducting ground) was derived by \( \text{Wait} \) [1998] and employed by \( \text{Rakov and Tinti} \) [2003] and \( \text{Thottappillil} \) and \( \text{Rakov} \) [2005]. That expression can be readily obtained from (7a) and (7b). A version of that equation, excluding the effect of ground, is given below.

\[
E_{\text{step}}(t) = \frac{1}{4\pi v} \frac{v \sin \theta}{1 - \frac{v}{c} \cos \theta} \int_0^{v(t-v/c)} \frac{1}{\lambda} \left( 1 - \frac{v}{c} \cos \theta \right) \cdot \frac{z'}{v} e^{v(t-v/c)z'} dt
\]

(9)

In section 3, we will show how application of the Duhamel’s integral \( [\text{Greenwood}, 1991] \) to (9) can be used to obtain equation (7a), representing the return stroke far field on the basis of the MTLE model for an arbitrary current \( i(0, t - r/c) \) at the channel base.

3. Comparison of Far Electric Field Equation Derived for the MTLE Model by \( \text{Shao et al.} \) [2005] With Equation (7a)

[10] \( \text{Shao et al.} \) [2005], henceforth denoted as SFJ, have presented an expression for far field (not accounting for ground effect) for the MTLE model that is reproduced below, after adapting it to the notation (see Figure 1) of this paper.

\[
E_{\text{farSFJ}} = \frac{1}{4\pi v} \frac{v \sin \theta}{1 - \frac{v}{c} \cos \theta} \cdot \left[ i\left(0, t - \frac{r}{c}\right) - \lambda \left(1 - \frac{v}{c} \cos \theta\right) v(t-v/c) \right. \\
\left. \cdot \int_0^{v(t-v/c)} i\left(0, t - \frac{r}{c} - \frac{z'}{v} \right) \cdot e^{v(t-v/c)z'} dz' \right]
\]

(10)

Equation (10) is to be compared with equation (7a). The first term in both equations represents the radiation field predicted by the TL model (see equation (6a)), and is the same. However, three differences between the second terms of (10) and (7a) can be observed. These are (1) SFJ have \( \lambda = \lambda'(1 - v \cos \theta/c) \) in two places, by multiplying factor before the integral and in the argument of the exponential function inside the integral, whereas (7a) has only \( \lambda \) in both these places. (2) In (10) the upper limit of integration \( L' = v(t - r/c) \) is the height of the return stroke channel at retarded time \( t - r/c \), whereas in (7a) it is given by (5b), the height of the return stroke channel contributing to the field at time \( t \), an approximation for \( L(t) \) valid at far distances. (3) In (7a) the return stroke current at height \( z' \) is given by \( i\left(0, t - \frac{R(z')}{c} - \frac{z'}{v}\right) \), whereas in (10) it is given by \( i(0, t - \frac{r}{c} - \frac{z'}{v}) \). For a field point above ground, \( r > R(z') \), and hence \( t - r/c - \frac{z'}{v} < (t - R(z')/c) - \frac{z'}{v} \). In spite of the apparent differences between (7a) and (10), they are equivalent, as will be shown both analytically and numerically. Let us first derive (10) starting from (7a).

[11] Far away from the channel, i.e., when \( z' \ll r, R(z') \approx r - z' \cos \theta \), and therefore

\[
t - \frac{z}{v} \approx t - \frac{z}{v} \left(1 - \frac{v}{c} \cos \theta\right) - \frac{r}{c}
\]

(11)

Let us define

\[
z' = \frac{z}{v} \left(1 - \frac{v}{c} \cos \theta\right) \equiv \frac{z}{v}
\]

(12)

From which

\[
z' = 1 - \left(\frac{v}{c} \cos \theta\right)\frac{z}{v}
\]

(13)

Equation (13) is similar in form to (5b). Let us now express \( z' \) and \( dz' \) in (7a) in terms of \( z \) and \( dz \). Then for the limits of the integral in (7a), when \( z' = 0, z = 0 \) and when \( z' = 1 - \left(\frac{v}{c} \cos \theta\right)\frac{z}{v} \), \( z = v(t - r/c) \). Thus (7a) can be written as

\[
E_{\text{farMTLL}} = \frac{1}{4\pi v} \frac{v \sin \theta}{1 - \frac{v}{c} \cos \theta} \cdot \left[ i(0, t - \frac{r}{c}) - \lambda \int_0^{v(t-r/c)} i(0, t - \frac{r}{c} - \frac{z}{v}) e^{v(t-v/c)z'} dz' \right]
\]

(14)

Rearranging (14), we get the same expression as (10), derived by SFJ. Note that \( z \) in (14) and \( z' \) in (10) are variables of integration that vary between the same limits of integration and hence are interchangeable.

[12] Thus we have shown that SFJ’s expression for far fields, equation (10), is equivalent to our expression (7a), even though two different methods of derivation are adopted in SFJ and in this paper. It should be pointed out that the upper limit of the integral in (10), \( v(t - r/c) \), gives the length of the return stroke channel at retarded time, not the length of the channel actually contributing to the field at a given time \( t \). On the other hand, the upper limit of the integral in (7a) gives the length of the channel actually contributing to the field at a given time \( t \). As an example, the length of the return stroke channel at a retarded time of one microsecond for a return stroke speed of 2.7 \( \times 10^8 \) m/s is 270 m, but the length of the channel contributing to the field at a far distance at angle \( \theta = 10^\circ \) will be 2375 m.

Therefore the appropriate far field condition is \( L(t) \ll r \), where \( L(t) \) is given by (5b), and not \( v(t - r/c) \ll r \), as given by \( \text{Shao et al.} \) [2004].

[13] Using a procedure similar to that employed in deriving (10) from (7a), one can get an expression that is different from but equivalent to (8a) for the MTLL model as

\[
E_{\text{farMTLL}} = \frac{1}{4\pi v} \frac{v \sin \theta}{1 - \frac{v}{c} \cos \theta} \cdot \left[ i(0, t - \frac{r}{c}) - \frac{1}{\lambda} - \int_0^{v(t-r/c)} i(0, t - \frac{r}{c} - \frac{z}{v}) e^{v(t-v/c)z'} dz' \right]
\]

(15)
Field expressions for image channel corresponding to (10) and (15) for the MTLE and MTLL models, can be derived from (7b) and (8b), respectively. In fact, one only needs to replace $1 - (v/c)\cos\theta$ in (10) and (15), wherever it appears, with $1 + (v/c)\cos\theta$. It is interesting to note that the upper limit of integration of the second terms in (10) and (15) remains unchanged in the expressions for image channel, whereas upper limits of integration in expressions (7b) and (8b) differ from those in (7a) and (8a).

4. Far Field Expressions Containing Time Integrals for the MTLE and MTLL Models

[14] In the following, we will derive expressions for the MTLE model containing the time integral and show that it is equivalent to expressions (7a) and (10) containing the spatial integral. In doing so, we will use the fact that both equations (7a) and (10) reduce to equation (9) for the special case of step function current at the channel base and Duhamel’s integral. If equation (9) is the electric field response of a step current input at the bottom of the channel, then one can get the field response for any other specified current at the channel base by applying the Duhamel’s integral, because we are dealing with a linear system. One form of Duhamel’s integral is given by [Greenwood, 1991]

$$E(r, t) = E_{\text{ustep}}(r/c) \cdot i(0, t - r/c) + \int_0^t \left[ \frac{dE_{\text{ustep}}}{d\tau} \right] i(t - \tau) d\tau$$

(16)

where $E(r, t)$ is the electric field at distance $r$ (response) due to a specified channel-base current $i(0, t - r/c)$ (input), $E_{\text{ustep}}$ is the field response to a unit step current at the channel base, $r/c$ is the initial time, $t$ is the present time, and $\tau$ is an arbitrary time between $r/c$ and $t$. The unit step response $E_{\text{ustep}}$ is given by equation (9) divided by $I_0$. From (16), we get,

$$E(r, t) = \frac{Fv\sin\theta}{4\pi\epsilon_0 c^2 r} \cdot \left[ i(0, t - \frac{r}{c}) - \frac{Fv}{H} \int_{r/c}^t i(0, t - \tau) d\tau \right]$$

(17)

where $F = [1 - (v/c)\cos\theta]^{-1}$. It is shown in Appendix C that (17) is analytically equivalent to (7a). It should be also equivalent to (10), since the equivalence between (7a) and (10) has been shown in section 3 above.

[15] Equivalence between (17), (7a), and (10) were also verified numerically, as illustrated in Figure 4. Note that (7a) and (10) contain spatial integrals (with different upper limits), while (17) contains a time integral. Nevertheless, fields predicted by these three equations are indistinguishable.

[16] An equation similar to (17) can be readily derived (substituting (C1)–(C6) of Appendix C in (8a)) for the MTLL model, the result being given by

$$E_{\text{farMTLL}}(r, t) = \frac{Fv\sin\theta}{4\pi\epsilon_0 c^2 r} \cdot \left[ i(0, t - \frac{r}{c}) - \frac{Fv}{H} \int_{r/c}^t i(0, t - \tau) d\tau \right]$$

(18)

Figure 5 shows, in a format similar to that of Figure 4, a comparison of three equivalent equations for the MTLL model. Again, as expected, an excellent agreement is seen.

5. Conclusions

[17] Far field approximate expressions at any elevation angle for the MTLE and MTLL return stroke models are derived and their validity is tested against exact solution. It is shown that different (for example, containing a spatial or...
time integral), but equivalent approximate expressions, can be derived for each of the considered models.

**Appendix A: Far Electric Field Expressions Based on the TL, MTLE, and MTLL Models (Effects of Ground Not Included)**

**A1. TL Model**

[18] The approximation for far field at an arbitrary elevation angle is found, for example, in the works of LeVine and Willett [1992], Krider [1992], and Thottappillil et al. [1998], and the resultant expression is equation (6a). This equation is valid whether there is current discontinuity at the upward moving front or not.

**A2. MTLE Model**

[19] Applying (3) to retarded current, the current derivative at height \( z' \) is given by

\[
\frac{\partial i(t', z' - v R(z')/c)}{\partial t} = \frac{\partial i(0, t - z'/v - R(z')/c) \cdot e^{-\zeta/\lambda}}{\partial z} = \frac{\partial i(0, t - z'/v - R(z')/c)}{\partial \tilde{z}} \cdot e^{-\zeta/\lambda} \tag{A1}
\]

The relationship between the time derivative of retarded current in (A1) and its spatial derivative is given by [e.g., Thottappillil et al., 1998]

\[
\frac{\partial i(0, t - z'/v - R(z')/c)}{\partial \tilde{z}} = -\frac{\partial i(0, t - z'/v - R(z')/c)}{\partial \tilde{z}} \cdot v \cdot F_{TL}(z') \tag{A2}
\]

where

\[
F_{TL}(z') = \frac{1}{1 - \frac{v}{c} \cos \alpha(z')} \tag{A3}
\]

is the so called F factor discussed extensively by several authors [e.g., Rubinstein and Uman, 1990; LeVine and Willett, 1992; Krider, 1992; Thottappillil et al., 1998; Shao et al., 2004; Thottappillil and Rakov, 2005]. The speed, \( \frac{dL'}{dt} \), of the wave front as “seen” by the observer at \( P \) (Figure 1) is given by Thottappillil et al. [1998, equation [31]] as

\[
\frac{dL'}{dt} = v \cdot \left( \frac{1}{1 - \frac{v}{c} \cos \theta(L')} \right) = v \cdot F_{TL}(L') \tag{A4}
\]

Substituting (A2) in (A1) and the resulting equation and (A4) in (5a), we get

\[
E_{MTLE}(r, \theta, t) = -\frac{1}{4 \pi \varepsilon_0} \int_0^{L''(t)} \alpha(z') \frac{\sin \alpha(z')}{c R(z')} \frac{\partial i(0, t - z' - R(z')/c)}{\partial \tilde{z}}
\cdot v \cdot F_{TL}(z') \cdot e^{-\zeta/\lambda} d\tilde{z}'
\]

\[
+ \frac{1}{4 \pi \varepsilon_0} \int_0^{L''(t)} v \cdot F_{TL}(L') \cdot \sin \alpha(L') \cdot i(0, 0) e^{-L''(t)/\lambda} \alpha(L')
\]

\[
\tag{A5}
\]

Expanding the product under the integral in (A6) by partial integration and simplifying we get equation (7a), the far field expression based on the MTLE model. Similar to equation (6a), equation (7a) is valid whether there is current discontinuity at the upward moving front or not.

**A3. MTLL Model**

[20] Using a procedure similar to that adopted above for the MTLE model, one can readily derive equation (8a), the far electric field equation based on the MTLL model, with or without discontinuity at the upward moving front.

**Appendix B: Exact Electric Field Equation (Effects of Ground Not Included)**

[21] The exact expression for remote electric field for an arbitrary current distribution along a vertical channel is presented by Thottappillil et al. [1998], which is reproduced as equation (B1) below. In the first and third terms of (B1), \( v \) appearing in the lower limits of time integrals is the return stroke speed. Other symbols used in (B1) are defined in Figure 1. The effect of ground plane is not included. The sum of the fifth and sixth terms in (B1) is the radiation field, which is given by equation (5a) in section 2.

\[
E(r, \theta, t) = \frac{1}{4 \pi \varepsilon_0} \int_0^{L''(t)} \left[ \frac{R(z')}{R^2(z')} \int_{\frac{t - R(z')}{c}}^{t} i(z', \tau - R(z')/c) d\tau \right] d\tilde{z}'
\]

\[
+ \frac{1}{4 \pi \varepsilon_0} \int_0^{L''(t)} \left[ \frac{R(z')}{R^2(z')} \int_{\frac{t - R(z')}{c}}^{t} i(z', \tau - R(z')/c) d\tau \right] d\tilde{z}'
\]

\[
+ \frac{1}{4 \pi \varepsilon_0} \int_0^{L''(t)} \left[ \frac{R(z')}{R^2(z')} \int_{\frac{t - R(z')}{c}}^{t} i(z', \tau - R(z')/c) d\tau \right] d\tilde{z}'
\]

\[
+ \frac{1}{4 \pi \varepsilon_0} \int_0^{L''(t)} \left[ \frac{R(z')}{R^2(z')} \int_{\frac{t - R(z')}{c}}^{t} i(z', \tau - R(z')/c) d\tau \right] d\tilde{z}'
\]

\[
+ \frac{1}{4 \pi \varepsilon_0} \int_0^{L''(t)} \left[ \frac{R(z')}{R^2(z')} \int_{\frac{t - R(z')}{c}}^{t} i(z', \tau - R(z')/c) d\tau \right] d\tilde{z}'
\]

\[
\tag{A6}
\]
\[
\begin{align*}
\frac{1}{4\pi\epsilon_0} \int_0^{L(t)} \left[ \frac{\alpha(z') \sin \alpha(z')}{c^2 R(z')} \frac{\partial}{\partial t} \left( t - \frac{R(z')}{c} \right) \right] dz' \\
+ \frac{1}{4\pi\epsilon_0} \alpha(L(t)) \frac{\sin \alpha(L(t))}{c^2 R(L(t))} \cdot \left( L(t) - \frac{R(L(t))}{c} \right) \frac{dL(t)}{dt}
\end{align*}
\]

(B1)

In equation (B1), \( dL(t)/dt \) is the speed of the current wavefront as "seen" by the observer at \( P \), which is different from the real speed \( v \). The lower limit of the time integral in the first and third terms of (B1) is the time at which the return stroke wavefront has reached the height \( z' \) for the first time, as "seen" from the observation point. The time \( \tau \) and the length of the channel contributing to the field at \( P \) are related by the equation \( \tau = \frac{z'}{v} + \frac{R(z')}{c} \). The sixth term of (B1) containing \( dL(t)/dt \) will have a nonzero value only if there is a current discontinuity (nonzero current) at the wavefront. Equation (B1) can be resolved into components explicitly in the directions of the spherical coordinate unit vectors \( \theta \) and \( \phi \), as done by Thottappillil and Rakov [2001a, 2001b, 2007]. At a distance of 100 km, the \( \phi \) component is negligible compared to the \( \theta \) component [e.g., Thottappillil and Rakov, 2007].

**Appendix C: Equivalence Between Equations (7a) and (17)**

Symbol \( \tau \) in (17) is the variable of integration, and symbol \( z' \) is the variable of integration in (7a). Therefore we have certain freedom in defining them, as long as they are consistent with each other. Let us define \( \tau \) in a physically meaningful way as the time it takes for a return stroke wavefront traveling at speed \( v \) to reach height \( z' \) plus the time it takes for a signal traveling from that height at the speed of light \( c \) to reach the field point at distance \( R(z') \).

\[
\tau = \frac{z'}{v} + \frac{R(z')}{c}
\]

(C1)

From (C1) we can get [e.g., Thottappillil et al., 1998]

\[
\frac{dz'}{d\tau} = \frac{v}{1 - (v/c) \cos \alpha(z')} = F(z') \cdot v
\]

(C2)

Therefore

\[
dz' = F(z') \cdot v \cdot d\tau
\]

(C3)

Far from the channel,

\[
F(z') \approx F = \frac{1}{1 - (v/c) \cos \theta}, \quad \text{and} \quad \alpha(z') \approx \theta
\]

(C4)

Also, when

\[
z' = 0, \quad \tau = r/c \quad \text{and} \quad \tau = t
\]

(C5)

Therefore from (C4) and (C5),

\[
z' = F \cdot v \cdot (\tau - r/c)
\]

(C6)

Applying (C1)–(C6) to (7a), we get (17).

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**References**


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V. A. Rakov, Department of Electrical and Computer Engineering, University of Florida, 553 Engineering Building 33, Gainesville, FL 32611-6130, USA.

N. Theethayi and R. Thottappillil, Division for Electricity and Lightning Research, Uppsala University, Box 534, 75121 Uppsala, Sweden.