JOURNAL OF GEOPHYSICAL RESEARCH, VOL. 102, NO. D6, PAGES 6987–7006, MARCH 27, 1997

# Distribution of charge along the lightning channel: Relation to remote electric and magnetic fields and to return-stroke models

Rajeev Thottappillil[1]
Institute of High Voltage Research, Uppsala University, Uppsala, Sweden

Vladimir A. Rakov[2] and Martin A. Uman
Department of Electrical and Computer Engineering, University of Florida, Gainesville

**Abstract:** We derive exact expressions for remote electric and magnetic fields as a function of the time- and height-varying charge density on the lightning channel for both leader and return-stroke processes. Further, we determine the charge density distributions for six return-stroke models. The charge density during the return-stroke process is expressed as the sum of two components, one component being associated with the return-stroke charge transferred through a given channel section and the other component with the charge deposited by the return stroke on this channel section. After the return-stroke process has been completed, the total charge density on the channel is equal to the deposited charge density component. The charge density distribution along the channel corresponding to the original transmission line (TL) model has only a transferred charge density component so that the charge density is everywhere zero after the wave has traversed the channel. For the Bruce-Golde (BG) model there is no transferred, only a deposited, charge density component. The total charge density distribution for the version of the modified transmission line model that is characterized by an exponential current decay with height (MTLE) is unrealistically skewed toward the bottom of the channel, as evidenced by field calculations using this distribution that yield (1) a large electric field ramp at ranges of the order of some tens of meters not observed in the measured electric fields from triggered-lightning return strokes and (2) a ratio of leader-to-return-stroke electric field at far distances that is about 3 times larger than typically observed. The BG model, the traveling current source (TCS) model, the version of the modified transmission line model that is characterized by a linear current decay with height (MTLL), and the Diendorfer-Uman (DU) model appear to be consistent with the available experimental data on very close electric fields from triggered-lightning return strokes and predict a distant leader-to-return-stroke electric field ratio not far from unity, in keeping with the observations. In the TCS and DU models the distribution of total charge density along the channel during the return-stroke process is influenced by the inherent assumption that the current reflection coefficient at ground is equal to zero, the latter condition being invalid for the case of a lightning strike to a well-grounded object where an appreciable reflection is expected from ground.

## Introduction

If the spatial and temporal distribution of either the current or the charge density along the lightning channel is known, the electromagnetic fields at any distance from the channel can be uniquely calculated from Maxwell's equations. Expressions relating the current distribution along a straight vertical lightning channel to the remote electric and magnetic fields have been derived by *Uman et al.* [1975] and are now commonly found in the literature [e.g., *Lin et al.*, 1980; *Uman*, 1987; *Nucci et al.*, 1990]. Here we derive similar general expressions relating the spatial and temporal distribution of charge density along the channel to the remote electric and magnetic fields. The equivalence between the expressions for the fields as a function of the channel current, traditionally applied to the lightning return stroke [e.g., *Lin et al.*, 1980; *Uman*, 1987; *Nucci et al.*, 1990], and the fields as a function of the charge density, traditionally applied in the "static" approximation to the lightning stepped and dart leader processes [e.g., *Schonland et al.*, 1938; *Uman*, 1987; *Rubinstein et al.*, 1995] is demonstrated. Additionally, we present an expression for the electrostatic field produced by a descending lightning leader in terms of both height- and time-varying charge density and compare this rigorous solution to the time-independent electrostatic approximation proposed by *Thomson* [1985]. It is important to note that the field equations in terms of charge density for the leader and return stroke can potentially be used for studying the link between these two processes. This is so because both the leader and return-stroke processes act on the same charge density (the return stroke

[1]Formerly at Department of Electrical and Computer Engineering, University of Florida, Gainesville.
[2]Also at High Voltage Research Institute, Tomsk Polytechnical University, Tomsk, Russia.



Paper number 96JD03344.
0148-0227/97/96JD-03344$09.00

transports the leader charge from the channel to ground), while currents associated with the two processes differ by an order of magnitude or so. The traditional return-stroke field equations in terms of channel current and leader field equations in terms of charge density serve to isolate the leader and return-stroke processes that are undoubtedly coupled. Further, we examine charge density distributions along the lightning channel associated with six return-stroke models, namely, the Bruce-Golde (BG), the transmission line (TL), two versions of the modified transmission line (the MTLL with a linear decay of current with height and the MTLE with an exponential decay), the traveling current source (TCS), and the Diendorfer-Uman (DU), and compare our results with those published by *Nucci et al.* [1990]. Finally, we test the validity of the charge density distributions associated with various models (and hence the validity of the models) using published data on close and distant lightning electric fields.

## Theory, Analysis, and Discussion

### Expressions for the Lightning Electric and Magnetic Fields in Terms of the Channel Charge Density

**General.** The differential vertical electric field $dE_z$ and the azimuthal magnetic field $dB_\phi$ at ground level due to a vertical, current-carrying channel element of differential length $dz'$ at height $z'$ above a perfectly conducting Earth and at horizontal distance $r$ from the observation point can be expressed in terms of current as [e.g., *Uman*, 1987]

$$dE_z(r,t) = \frac{dz'}{2\pi\epsilon_0}\left[\frac{2z'^2-r^2}{R^5(z')}\int_{t_b(z')}^{t} i\left(z',\tau-\frac{R(z')}{c}\right)d\tau + \frac{2z'^2-r^2}{cR^4(z')}i\left(z',t-\frac{R(z')}{c}\right) - \frac{r^2}{c^2R^3(z')}\frac{\partial i(z',t-R(z')/c)}{\partial t}\right] \tag{1}$$

$$dB_\phi(r,t) = \frac{dz'}{2\pi\epsilon_0 c^2}\cdot\left[\frac{r}{R^3(z')}i\left(z',t-\frac{R(z')}{c}\right) + \frac{r}{cR^2(z')}\frac{\partial i(z',t-R(z')/c)}{\partial t}\right] \tag{2}$$

where $t_b(z')$ is the time at which the current is "seen" by an observer at $P$ to begin in the channel section at height $z'$ (see Figure 1), $c$ is the speed of light in vacuum, and $R(z') = (z'^2+r^2)^{1/2}$. The total fields are found by integrating (1) and (2) over the contributing channel length. Two particularly useful specific applications of the general fields (1) and (2) are to the case of return stroke propagating upward from ground level and the case of a leader process propagating downward from a spherically symmetrical cloud charge source. Using the continuity equation relating the current and charge density, we can express, as shown in Appendix B, the electric and magnetic fields of the return-stroke and leader processes as a function of the channel charge per unit length (line charge density) $\rho_L(z',t)$. In the following, using the material in Appendix B, we introduce equations written as a function of the line charge density for the total electric and magnetic fields of the return stroke and for the electrostatic and magnetostatic fields of the leader, with the total leader fields being found in Appendix B.

**Return stroke.** The return stroke is assumed to create an extending channel whose lower end is fixed at ground and whose upper end is associated with the return-stroke front that moves from ground ($z' = 0$) upward with a constant speed $v$. The observer at $P$ "sees" the return-stroke front passing a height $z'$ at time $t_b(z')=z'/v + R(z')/c$ (see Figure 1a). Thus the "radiating" length $H(t)$ of the channel, that is, the length traversed by the upward moving front as "seen" by the observer at time $t$, is given by the solution of

$$t = \frac{H(t)}{v} + \frac{(H^2(t)+r^2)^{1/2}}{c} \tag{3}$$

For the case where there is no current discontinuity at the return- stroke front, the total electric and magnetic fields are obtained by integrating the differential electric and magnetic fields along the channel from 0 to $H(t)$, as described in section B4 of Appendix B,

$$\begin{aligned}E_z(r,t) = &-\frac{1}{2\pi\epsilon_0}\int_0^{H(t)}\frac{z'}{R^3(z')}\rho_L\left(z',t-\frac{R(z')}{c}\right)dz' \\ &-\frac{1}{2\pi\epsilon_0}\int_0^{H(t)}\left(\frac{3}{2}\frac{z'}{cR^2(z')}-\frac{1}{2}\frac{\tan^{-1}(z'/r)}{cr}\right)\cdot\frac{\partial\rho_L[z',t-R(z')/c]}{\partial t}dz' \\ &-\frac{1}{2\pi\epsilon_0}\int_0^{H(t)}\frac{z'}{c^2R(z')}\frac{\partial^2\rho_L[z',t-R(z')/c]}{\partial t^2}dz' \\ &-\frac{1}{2\pi\epsilon_0}\left(\frac{3}{2}\frac{H(t)}{cR^2(H(t))}-\frac{1}{2}\frac{\tan^{-1}[H(t)/r]}{cr}\right)\cdot\rho_L\left(H(t),\frac{H(t)}{v}\right)\frac{dH(t)}{dt} \\ &-\frac{1}{2\pi\epsilon_0}\frac{H(t)}{c^2R[H(t)]}\frac{\partial}{\partial t}\left[\rho_L\left(H(t),\frac{H(t)}{v}\right)\frac{dH(t)}{dt}\right]\end{aligned} \tag{4}$$

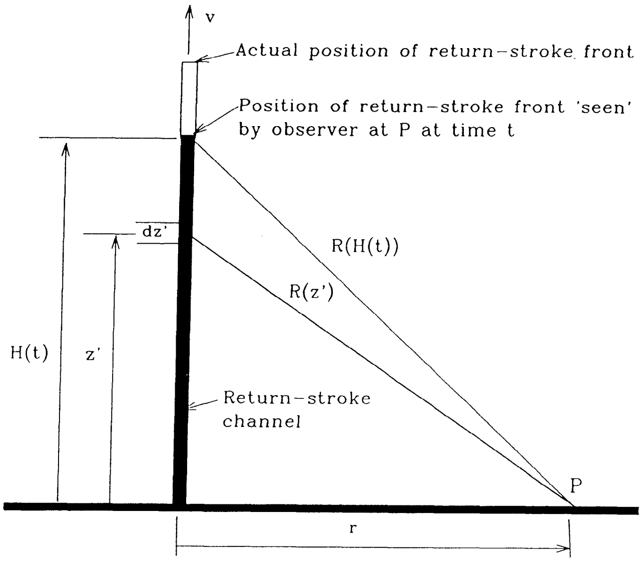


**Figure 1a.** Geometry used in deriving the expressions for electric and magnetic fields at a point P on Earth a horizontal distance $r$ from the vertical lightning return-stroke channel extending upward with speed $v$.

$$B_\phi(r,t) = \frac{1}{2\pi\epsilon_0 c^2} \cdot \int_0^{H(t)} \frac{1}{r}\frac{z'}{R(z')}\frac{\partial\rho_L[z',t-R(z')/c]}{\partial t}dz' + \frac{1}{2\pi\epsilon_0 c^2}\int_0^{H(t)}\frac{1}{c}\tan^{-1}\left(\frac{z'}{r}\right) \cdot \frac{\partial^2\rho_L[z',t-R(z')/c]}{\partial t^2}dz' + \frac{1}{2\pi\epsilon_0 c^2}\frac{1}{r}\frac{H(t)}{R[H(t)]}\rho_L\left(H(t),\frac{H(t)}{v}\right)\frac{dH(t)}{dt} + \frac{1}{2\pi\epsilon_0 c^2}\frac{1}{c}\tan^{-1}\left(\frac{H(t)}{r}\right) \cdot \frac{\partial}{\partial t}\left[\rho_L\left(H(t),\frac{H(t)}{v}\right)\frac{dH(t)}{dt}\right] \tag{5}$$

The electrostatic field component is represented by the first term of (4), the induction electric field component by the second and fourth terms and the radiation electric field component by the third and fifth terms. The magnetostatic field and radiation magnetic field components are represented by the first and third, and the second and fourth terms of (5), respectively. Note that in (4) and (5) there are two terms for the induction and for the radiation components rather than the one term found in the equivalent formulations (see, for instance, discussion of (7.3) and (7.4) of *Uman* [1987]) in terms of channel current. If there is a current discontinuity at the return-stroke front, the electric and magnetic fields due to the discontinuity given by (B31) and (B32), derived in section B5 of Appendix B, should be added to (4) and (5), respectively, bringing to three the number of terms for each of the radiation field components in the charge formulations.

In the case that the return-stroke speed is an arbitrary function of height, (4) and (5) are valid if the constant speed $v$ is replaced by the "average" speed defined by *Thottappillil et al.* [1991].

We computed electric and magnetic fields at distances of 0.05, 1, and 100 km using (4) and (5) with (B31) and (B32) when applicable for the six return-stroke models described in section 2 below. As expected, the results were found to be identical to those obtained using the traditional field formulations [e.g., *Uman*, 1987, Equations (7.3) and (7.4)] in terms of channel current.

**Leader.** The leader is assumed to create a channel extending vertically downward with a constant speed $v$ from a stationary and spherically symmetrical charge source at height $H_m$ (see Figure 1b). At time $t$, the observer "sees" the lower end of the leader channel at a height $h(t)$ given by the solution of

$$t = \frac{H_m - h(t)}{v} + \frac{\sqrt{h^2(t)+r^2}}{c} \tag{6}$$

The total leader electric and magnetic fields can be found by integrating the corresponding differential fields from $h(t)$ to $H_m$ including those of the charge source at $H_m$, as described in section B6 of Appendix B, and are given by (B38) and (B39), respectively.

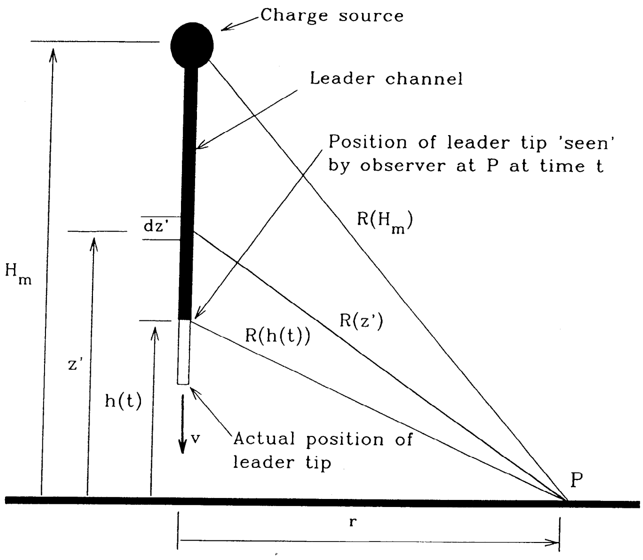


**Figure 1b.** Geometry used in deriving the expressions for electric and magnetic fields at a point P on Earth a horizontal distance r from the vertical lightning leader channel extending downward with speed $v$.

Here, as an example of the application of the leader field expressions in terms of charge density, we consider the electrostatic approximation for the leader electric field and magnetostatic approximation for the leader magnetic field, the approximations expected to be applicable to close lightning. Retaining only the first and fourth terms in (B38), we get

$$E_z(r,t) = \frac{1}{2\pi\epsilon_0}\int_{h(t)}^{H_m}\frac{z'}{R^3(z')}\rho_L\left(z',t-\frac{R(z')}{c}\right)dz' - \frac{1}{2\pi\epsilon_0}\frac{H_m}{R^3(H_m)}\int_{h(t)}^{H_m}\rho_L\left(z',t-\frac{R(z')}{c}\right)dz' \tag{7}$$

where $R(H_m)=(H_m^2+r^2)^{1/2}$ (see Figure 1b). The first term of (7) represents the field change due to the charge on the leader channel, and the second term represents the field change due to the depletion of the charge at the cloud charge source as it is drained by the extending leader channel. The total charge on the leader channel at any time is equal to the total charge drained by that time from the cloud charge source. If the maximum difference in propagation times from sources on the channel to the observer is much less than the time required for significant variation in the sources, we can rewrite (7) as

$$E_z(r,t) = \frac{-1}{2\pi\epsilon_0}\int_{H_m}^{z_t}\left[\frac{z'}{R^3(z')} - \frac{H_m}{R^3(H_m)}\right]\rho_L(z',t)dz' \tag{8}$$

where $z_t=H_m-vt$ is the height of the leader tip at time $t$ and $v$ is the leader speed, assumed to be a constant. *Thomson* [1985] has derived a similar equation (his equation (5)) based on Coulomb's law, although he assumes that the charge density distribution on the leader channel behind leader tip does not vary with time. Thomson's assumption is equivalent to the assumption of a uniform current in the channel between the cloud charge source and leader tip, as follows from the continuity equation (A5b). In this respect, Thomson's leader model is similar to the BG model for return strokes, while (8) permits any other (e.g., a TCS type) leader model.

The magnetostatic approximation for the leader magnetic field is obtained from (B39) by neglecting the second, fifth, and sixth terms associated with the radiation field

$$
\begin{aligned}
B_\phi(r,t) = {} & \frac{-1}{2\pi\epsilon_0 c^2}\frac{1}{r}\int_{h(t)}^{H_m}\frac{z'}{R(z')} \\
& \cdot \frac{\partial\rho_L\left(z',t-\frac{R(z)}{c}\right)}{\partial t}dz' \\
& + \frac{1}{2\pi\epsilon_0 c^2}\left[\frac{1}{r}\frac{H_m}{R(H_m)}\frac{d}{dt}\int_{h(t)}^{H_m}\rho_L\left(z',t-\frac{R(z')}{c}\right)dz'\right] \\
& + \frac{1}{2\pi\epsilon_0 c^2}\left[\frac{1}{r}\frac{h(t)}{R[h(t)]}\rho_L\left(h(t),t-\frac{R[h(t)]}{c}\right)\frac{dh(t)}{dt}\right]
\end{aligned}
\tag{9}
$$

Neglecting retardation effects, applying the Leibnitz's formula given by (B5) to the second term, replacing $h(t)$ by $z_t$ and $dh(t)/dt$ by $v$, the constant leader speed, and rearranging the terms, we can simplify (9) as

$$
\begin{aligned}
B_\phi(r,t) = {} & \frac{1}{2\pi\epsilon_0 c^2}\frac{1}{r}\int_{H_m}^{z_t}\left[\frac{z'}{R(z')}-\frac{H_m}{R(H_m)}\right] \\
& \cdot\frac{\partial\rho_L(z',t)}{\partial t}dz' \\
& + \frac{1}{2\pi\epsilon_0 c^2}\frac{1}{r}\left[\frac{z_t}{R(z_t)}-\frac{H_m}{R(H_m)}\right]\rho_L(z_t,t)v
\end{aligned}
\tag{10}
$$

Assuming that the leader charge density behind the leader tip does not vary with time ($\partial\rho_L(z',t)/\partial t = 0$), the approximation used by *Thomson* [1985], noting that $\rho_L(z_t,t)\,v = -\,i(z_t,t)$, that $c^2 = (\mu_0\varepsilon_0)^{-1}$, and that the continuity equation behind the leader tip requires $\partial i(z',t)/\partial z' = 0$ if $\partial\rho_L(z',t)/\partial t = 0$, we get for $z_t = 0$ (a fully developed leader channel),

$$
B_\phi\left(r,\frac{H_m}{v}\right) = \frac{\mu_0}{2\pi r}\frac{H_m}{R(H_m)}i\left(\frac{H_m}{v}\right) \tag{11}
$$

Equation (11) is the familiar expression [e.g., *Uman*, 1987] for the magnetostatic field of a vertical current-carrying line the bottom end of which is at ground and the top end is at height $H_m$. If the observation point is very close to the channel base so that $r << H_m$, (11) can be further simplified to give

$$
B_\phi(r,H_m/v) = \frac{\mu_0 i(H_m/v)}{2\pi r} \tag{12}
$$

which is the same equation as that for an infinitely long current-carrying line [e.g., *Sadiku*, 1994].

## Charge Distributions Along the Lightning Channel for Six Return-Stroke Models and Validation of the Models

**General: Continuity equation applied to return strokes.** The charge density at any height $z'$ on a straight vertical lightning channel at any time $t'$ is given in Appendix A by (A8), which is reproduced in (13) with $t'$ replaced by $t$,

$$
\rho_L(z',t) = \frac{i(z',z'/v)}{v} - \int_{z'/v}^{t}\frac{\partial i(z',\tau)}{\partial z'}d\tau \tag{13}
$$

It will be shown later by applying (13) to different return-stroke models, that in general the total charge density can be decomposed into two components, one being associated with the return-stroke charge transferred through a given channel section (a component that vanishes after the return stroke has been completed), and the other being associated with the charge deposited by the return stroke on this channel section (a nonzero component both during and after the return stroke). In the special cases of the TL and BG models one of the two components is absent at all times. The first term of (13) represents only the deposited charge density component, while the second term can contribute to both transferred and deposited charge density components.

Applying Leibnitz's formula to (13) we obtain,

$$
\begin{aligned}
\rho_L(z',t) &= -\frac{d}{dz'}\int_{z'/v}^{t} i(z',\tau)d\tau \\
&= -\frac{dQ(z',t)}{dz'}
\end{aligned}
\tag{14}
$$

where $Q(z',t)$ is the charge in the channel section at height $z'$ at time $t$. Equation (14), and hence (13), is equivalent to (3) of *Nucci et al.* [1990, p. 20,398] who used it to determine "the charge removed from the leader channel by the return stroke" up to time $t$ for various return-stroke models. It will be shown below that (13) determines the total charge density which is not necessarily equal in magnitude to the charge "removed" from the channel. The sign convention throughout this paper is that the return-stroke current which transports positive charge upward (or negative charge downward) is positive.

It is important to note that the charge density expressions derived in this section must be modified to take into account retardation effects if they are to be used in (4) and (5) for finding the remote electric and magnetic fields. To find the retarded charge densities (charge densities "seen" by the observer) from retarded currents for the various return stroke models, (B4) must be used instead of (13).

Since $z'/v$ is the time for the return stroke front to reach height $z'$, the first term of (13) is nonzero only if there is a current discontinuity at the front, an inherent feature of the BG and TCS models. The second term becomes zero if there is no current variation along the channel, as is the case for the BG model.

In the following, using (13), we examine the charge density, as a function of both time and height, predicted by six return-stroke models. A comparison with the results of a similar study presented by *Nucci et al.* [1990] is made for the models examined both here and by them. Additionally, we test the validity of the charge distributions associated with various models using the published experimental data on electric field waveforms produced by very close triggered-lightning return strokes and on leader-to-return-stroke electric field change ratios for natural-lightning return strokes at far ranges.

**Transmission line type models.** For models of the transmission-line type [*Uman and McLain*, 1969; *Rakov and Dulzon*, 1987; *Nucci et al.*, 1988] current as a function of $z'$ and $t$ is expressed by *Rakov and Dulzon* [1991] as

$$i(z',t) = P(z')\,i(0,t-z'/v) \tag{15}$$

where $P(z')$ is an arbitrary spatial attenuation function for the upward propagating current wave. For the original transmission line model [*Uman and McLain*, 1969], $P(z')=1$. If we take the spatial derivative of (15) with t replaced by $\tau$, we obtain

$$\begin{aligned}\frac{\partial i(z',\tau)}{\partial z'} &= \frac{\partial}{\partial z'}[P(z')\,i(0,\tau-z'/v)] \\ &= P(z')\frac{\partial i(0,\tau-z'/v)}{\partial z'} + \frac{dP(z')}{dz'}i(0,\tau-z'/v)\end{aligned} \tag{16}$$

Converting the spatial derivative of current to the time derivative of current via the chain rule, we get

$$\begin{aligned}\frac{\partial i(0,\tau-z'/v)}{\partial z'} &= \frac{\partial i(0,\tau-z'/v)}{\partial(\tau-z'/v)}\frac{\partial(\tau-z'/v)}{\partial z'} \\ &= -\frac{1}{v}\frac{\partial i(0,\tau-z'/v)}{\partial \tau}\end{aligned} \tag{17}$$

Thus (16) becomes

$$\frac{\partial i(z',\tau)}{\partial z'} = -\frac{P(z')}{v}\frac{\partial i(0,\tau-z'/v)}{\partial \tau} + \frac{dP(z')}{dz'}i(0,\tau-z'/v) \tag{18}$$

Further, using (15) with $t = z'/v$, we can write

$$\frac{i(z',z'/v)}{v} = P(z')\frac{i(0,0)}{v} \tag{19}$$

Substituting (18) and (19) in (13), and simplifying, we find,

$$\begin{aligned}\rho_L(z',t) &= P(z')\frac{i(0,0)}{v} + \left[P(z')\frac{i(0,\tau-z'/v)}{v}\right]_{z'/v}^{t} \\ &\quad - \frac{dP(z')}{dz'}\int_{z'/v}^{t} i(0,\tau-z'/v)d\tau \\ &= P(z')\frac{i(0,t-z'/v)}{v} - \frac{dP(z')}{dz'}\int_{z'/v}^{t} i(0,\tau-z'/v)d\tau\end{aligned} \tag{20}$$

Note that the term associated with the return-stroke front (the first term of (13)) drops out of the equation, so that (20) is valid regardless of whether or not there is a current discontinuity, that is, $i(0,0) \neq 0$, at the return-stroke front. Both terms of (20) resulted from the second term of (13). The first term of (20) represents the charge density associated with the propagating current wave. An increase of speed causes a decrease of this component of charge density. The second term of (20) is associated with the variation of current amplitude with height as specified by the attenuation function $P(z')$. Without attenuation, that is, $P(z')=1$ as in the original TL model, there is no current amplitude variation with height, and the second term becomes zero. When the current decays to zero over the entire channel, the first term of (20) is zero, and the second term equals the charge per unit length absorbed by the channel (spent by the return stroke in neutralizing the charge previously stored on the channel by the leader). The negative of the charge absorbed by the channel may be viewed as the leader charge removed from the channel. One can visualize the first term of (20) as being associated with the return-stroke charge transferred through a given channel section upward and the second term as being due to charge deposited by the return stroke on this channel section. We will use the concept of "transferred" (nonzero only when the current flows) and "deposited" (nonzero both when the current flows and after the current ceases to flow) return-stroke charge for the BG, TCS, and DU models as well (see Table 1). Note that in the simplest models, TL and BG, one of the two charge density components is absent at all times. *Nucci et al.* [1990] presented equations for the deposited charge density component for the BG, TCS, and MTLE return-stroke models.

**Case 1 (TL): $P(z') = 1$ [*Uman and McLain*, 1969]:** The current wave suffers no distortion and no attenuation while propagating along the channel,

$$i(z',t) = i(0,t-z'/v) \tag{21}$$

Using (20), we find

$$\rho_L(z',t) = \frac{i(0,t-z'/v)}{v} \tag{22}$$

When current ceases to flow in all channel sections of interest $\rho_L(z',t)=0$; that is, there is no deposited charge in this model. As a result, there is no charge density equation given by *Nucci et al.* [1990] for the TL model.

**Case 2 (MTLL): $P(z') = 1-z'/H$ [*Rakov and Dulzon*, 1987]:** The current wave suffers no distortion, but its amplitude decays linearly with height,

$$i(z',t) = \left(1-\frac{z'}{H}\right)i(0,t-z'/v) \tag{23}$$

where $H$ is the total length of the channel. This model was developed assuming a uniform distribution of leader charge along the channel as shown by *Rakov and Dulzon* [1991] and confirmed below. Using (20), we obtain

$$\begin{aligned}\rho_L(z',t) &= \frac{1}{v}\left(1-\frac{z'}{H}\right)i(0,t-z'/v) \\ &\quad + \frac{1}{H}\int_{z'/v}^{t} i(0,\tau-z'/v)d\tau \\ &= \left(1-\frac{z'}{H}\right)\frac{i(0,t-z'/v)}{v} + \frac{Q(z',t)}{H}\end{aligned} \tag{24}$$

At a time when the current along the entire channel decays to zero, the first term, representing transferred charge, of (24) becomes zero and the second term, representing deposited charge, approaches a constant value, corresponding to a uniform distribution of return-stroke charge absorbed by the channel, which implies a uniform distribution of the leader charge previously stored on the channel (assuming that the return stroke neutralizes all leader charge in the channel).

**Case 3 (MTLE): $P(z') = e^{-z'/\lambda}$ [*Nucci et al.*, 1988]:** Similar to case 2, but with an exponential decay of current amplitude with height,

$$i(z',t) = e^{-z'/\lambda} i(0,t-z'/v) \tag{25}$$

where $\lambda$ is the current decay constant (assumed by *Nucci et al.*, [1990] to be 2000 m). Using (20), we obtain,

$$\rho_L(z',t) = e^{-z'/\lambda}\frac{i(0,t-z'/v)}{v} + \frac{e^{-z'/\lambda}}{\lambda}Q(z',t) \tag{26}$$

The first term of (26) represents transferred charge and the second term, given also by (18) of *Nucci et al.* [1990], represents the return-stroke charge deposited on the channel. As the current along the entire channel ceases to flow, the first term of (26) becomes zero, and $Q(z',t)$ approaches a constant value, resulting in an exponential distribution of charge density along the lightning channel with its maximum at ground level.

**Bruce-Golde model [*Bruce and Golde*, 1941].** In this model at any instant of time, a uniform current is assumed to exist in the channel between ground and the return stroke front. For a uniform current to flow along the channel the continuity equation (A5b) requires a time-independent charge density, a feature shown to be the case below. This model can be viewed as a special case of the traveling current source model, discussed below, when $c$ is replaced by $\infty$.

For the BG model,

$$i(z',t) = i(0,t) \tag{27}$$

Thus

$$\frac{\partial i(z',\tau)}{\partial z'} = \frac{\partial i(0,\tau)}{\partial z'} = 0 \tag{28}$$

$$\frac{i(z',z'/v)}{v} = \frac{i(0,z'/v)}{v} \tag{29}$$

Substituting (28) and (29) in (13), we obtain

$$\rho_L(z',t) = \frac{i(0,z'/v)}{v} \tag{30}$$

which represents the return-stroke charge deposited on the channel by the moving front. There is no transferred charge in the BG model. As a result, (30) is the same as the equation given for the BG model by *Nucci et al.* [1990]. The charge density distribution versus height along the channel for the BG model has the same shape as the current versus time at ground level.

**Traveling current source model [*Heidler*, 1985].** In this model the current is assumed to be generated at the upward moving return-stroke front and to propagate to ground at the speed of light. Current at the front turns on instantaneously. As opposed to the transmission line type models, this model (as well as the DU model discussed below) requires an assumption on the conditions at the attachment point between the lightning channel and ground. *Heidler* [1985] assumes that there is no impedance discontinuity at the attachment point.

For the TCS model,

$$i(z',t) = i(0,t+z'/c) \tag{31}$$

Then

$$\begin{aligned}\frac{\partial i(z',\tau)}{\partial z'} &= \frac{\partial i(0,\tau+z'/c)}{\partial(\tau+z'/c)}\frac{\partial(\tau+z'/c)}{\partial z'}\\ &= \frac{1}{c}\frac{\partial i(0,\tau+z'/c)}{\partial\tau}\end{aligned} \tag{32}$$

$$\frac{i(z',z'/v)}{v} = \frac{i(0,z'/v+z'/c)}{v} \tag{33}$$

Substituting (32) and (33) in (13), we find

$$\begin{aligned}\rho_L(z',t) &= \frac{i(0,z'/v+z'/c)}{v} - \frac{1}{c}i(0,\tau+z'/c)\Big|_{z'/v}^{t}\\ &= \frac{1}{v}i\left(0,\frac{z'}{v}+\frac{z'}{c}\right) - \frac{1}{c}i(0,t+z'/c) + \frac{1}{c}i\left(0,\frac{z'}{v}+\frac{z'}{c}\right)\\ &= -\frac{i(0,t+z'/c)}{c} + \frac{i(0,z'/v^*)}{v^*}\end{aligned} \tag{34}$$

where $v^* = v/(1+v/c)$ is the apparent speed of the front. The first term, representing transferred charge, is associated with the downward propagating current wave, while the second term is due to the upward-moving front and represents the return-stroke charge deposited on the channel (different signs signify the different propagation directions; current in both terms is positive according to the sign convention stated above). If $c$ is replaced by $\infty$, the transferred charge density component is equal to zero and the TCS model reduces to the BG model. The second term of (34) is also given by (15) of *Nucci et al.* [1990] Note that at the front $\rho_L(z', z'/v) = i(0,z'/v^*)/v$, as evident from the second form of the right-hand side of (34), and is always positive (to neutralize the negitive leader charge), while at ground ($z' = 0$) $\rho_L(0,t) = -i(0,t)/c$ and is always negative. The latter statement is only true if there is no discontinuity at $t = 0$ in the channel-base current; otherwise, the charge density at ground has an additional (positive) component, $i(0,0)/v^*$, due to that discontinuity. If $i(0,0) = 0$, then the distribution of the charge density along the channel for the TCS model is bipolar during the return-stroke process. After the return stroke is complete ($t \to \infty$) the charge density at ground level is zero, as for the BG model. The presence of appreciable negative charge at the channel base during the return stroke, $\rho_L(0,t) = -i(0,t)/c$, follows from the assumed form of the current in the TCS model which implies that the reflection coefficient for current at ground is equal to zero; that is, the rate of removal of the charge from the channel bottom is equal to the rate of supply of charge to the bottom from the upper channel sections. This assumption is clearly invalid for well-grounded objects for which one expects a voltage reflection coefficient at ground close to -1; in this case, the charge density at ground would be forced to be close to zero at all times while the current, experiencing a reflection coefficient of +1 or so, should double. *Heidler and Hopf* [1994] attempted to modify the TCS model to take into account the wave reflections at ground and at the upward moving front, but in doing so they could not use the channel-base current as an input to the model and had to arbitrarily specify the traveling-current-source current instead. Thus, the original TCS model appears to be unrealistic (unless the grounding impedance at the strike point is comparable to the lightning channel impedance), whereas the modified TCS model of *Heidler and Hopf* [1994] with ground and front reflections does not belong to the family of models with specified channel-base current discussed here.

**Comparison to *Nucci et al.* [1990].** The equations for charge density given by *Nucci et al.* [1990] for the MTLE and TCS models represent only the deposited charge density component and therefore cannot generally be used for calculating electric and magnetic fields. The total charge density for the MTLE and TCS models is given by, respectively, (26) and (34) above. Further, as noted earlier, in the TL model the charge on the channel is never zero during the return-stroke process. It is only zero after all current has ceased to flow in the channel. The distribution of charge density during the return-stroke process for the TL model is given by

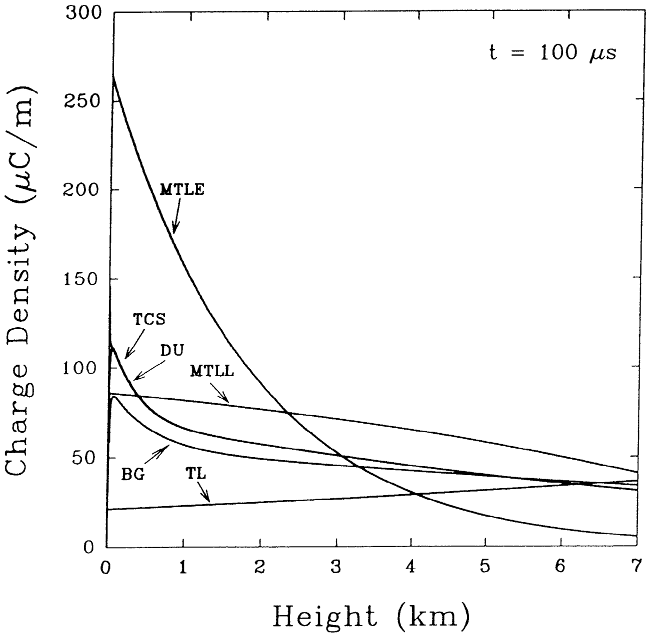


**Figure 2.** Calculated total charge density distributions for six return-stroke models at $t = 100$ μs. These distributions should be compared with the deposited charge density distributions given in Figure 7 of *Nucci et al.* [1990]. The current waveform shown in Figure 4a of *Nucci et al.* [1990] is assumed at ground level for all models. The speed of the return stroke is $1.3\times10^8$ m/s. The current decay constant $\lambda$ for the MTLE model is 2 km, and the total length H of the lightning channel for the MTLL model is 7.5 km. The discharge time constant for the DU model is 0.1 μs.

(22) of this paper. Figure 2 (to be compared with Figure 7 of *Nucci et al.*) shows the total charge density $\rho_L(z',t)$, as determined by (22), (26), (30), and (34) at $t$=100 μs using the current at the channel base adopted by *Nucci et al.* [1990, Figure 4a]. Additionally shown in Fig. 2 are $\rho_L(z',t)$ for two models not considered by Nucci et al.: the MTLL model (equation (24)) and the Diendorfer-Uman model discussed next. Note that Nucci et al. [1990] did not use their incomplete (except for the BG model) charge density equations to calculate fields; they used field equations written in terms of channel current instead.

**Diendorfer-Uman model [*Diendorfer and Uman*, 1990].** This model is similar to the TCS model, but current at the return-stroke front turns on exponentially with a time constant $\tau_D$. According to the DU model,

$$i(z',t) = i(0,t+z'/c) - i(0,\frac{z'}{v}+\frac{z'}{c})e^{-(t-z'/v)/\tau_D} \equiv i_1(z',t) - i_2(z',t) \tag{35}$$

If $\tau_D = 0$ the DU model reduces to the TCS model, and if, in addition to that, $c$ is replaced by $\infty$ we get the BG model. The line charge density associated with the first component of current in (35), $i_1(z',t)$, is the same as that for the TCS model and is given by (34). For the second component of current in (35),

$$\frac{\partial i_2(z',\tau)}{\partial z'} = \frac{\partial}{\partial z'}\left[i(0,\frac{z'}{v}+\frac{z'}{c})e^{-(\tau-z'/v)/\tau_D}\right] = \left[\frac{1}{v\tau_D}i(0,\frac{z'}{v}+\frac{z'}{c}) + \frac{\partial i(0,\frac{z'}{v}+\frac{z'}{c})}{\partial z'}\right]e^{-(\tau-z'/v)/\tau_D} \tag{36}$$

Further, setting $t = z'/v$ in the second component of current in (35), we find that

$$\frac{i_2(z',z'/v)}{v} = \frac{i(0,\frac{z'}{v}+\frac{z'}{c})}{v} \tag{37}$$

Defining $t^*(z') = z'/v + z'/c$ to be used in (36) and (37) and substituting (36) and (37) in (13), we obtain, the line charge density associated with $i_2(z',t)$

$$\rho_2(z',t) = \frac{i[0,t^*(z')]}{v} - \int_{z'/v}^{t} e^{-(\tau-z'/v)/\tau_D} \cdot \left[\frac{1}{v\tau_D}i[0,t^*(z')] + \frac{\partial i[0,t^*(z')]}{\partial z'}\right]d\tau \tag{38}$$

Although $t^*(z')$ is not a function of $\tau$, which is an arbitrary time between the limits $z'/v$ and $t$, we can write $t^*(z') = \tau - k$, where $k$ is a constant [*Diendorfer and Uman*, 1990]. Then,

$$\frac{\partial i[0,t^*(z')]}{\partial z'} = \frac{1}{v^*}\frac{\partial i[0,t^*(z')]}{\partial \tau} \tag{39}$$

where $v^*$ is same as in the TCS model above. Substituting (39) in (38) and evaluating the integral, we find,

$$\rho_2(z',t) = \frac{i[0,t^*(z')]}{v} - \frac{1}{v\tau_D} \cdot i(0,t^*(z'))\left[\frac{e^{-(\tau-z'/v)/\tau_D}}{-1/\tau_D}\right]_{z'/v}^{t} - \frac{1}{v^*}\frac{\partial i(0,t^*(z'))}{\partial t}\left[\frac{e^{-(\tau-z'/v)/\tau_D}}{-1/\tau_D}\right]_{z'/v}^{t}$$

$$= e^{-(t-z'/v)/\tau_D}\left[\frac{i(0,t^*(z'))}{v} + \frac{\tau_D}{v^*}\frac{\partial i(0,t^*(z'))}{\partial t}\right] - \frac{\tau_D}{v^*}\frac{\partial i(0,t^*(z'))}{\partial t} \tag{40}$$

Combining (40) and (34) and noting that $t^*(z') = z'/v^*$, we obtain the total charge density for the DU model as

$$\rho_L(z',t) = -\frac{i(0,t+z'/c)}{c} - e^{-(t-z'/v)/\tau_D} \cdot \left[\frac{i(0,z'/v^*)}{v} + \frac{\tau_D}{v^*}\frac{di(0,z'/v^*)}{dt}\right] + \frac{1}{v^*}\left[i(0,z'/v^*) + \tau_D\frac{di(0,z'/v^*)}{dt}\right] \tag{41}$$

At the front, $\rho_L(z',z'/v) = 0$, similar to the transmission line type models (provided that there is no discontinuity at $t = 0$ in the channel-base current) but in contrast with the BG and TCS models. At ground, the charge density depends on the time derivative of the channel-base current at $t = 0$, being identical to the charge density for the TCS model if this derivative is equal to zero. Discussion regarding the implicit assumption in the TCS model that the current reflection coefficient at ground

**Table 1.** Charge Density at Height $z'$ at Time $t$ for Various Lightning Return-Stroke Models

| Return-Stroke Model | $\rho_L(z',t)$ Transferred Charge Density Component | $\rho_L(z',t)$ Deposited Charge Density Component | $\rho_L(0,t_{RS})$ |
|---|---|---|---|
| TL | $\frac{i(0,t-z'/v)}{v}$ | - | - |
| MTLL | $\left(1-\frac{z'}{H}\right)\frac{i(0,t-z'/v)}{v}$ | $\frac{Q(z',t)}{H}$ | $\frac{Q(0,t_{RS})}{H}$ |
| MTLE | $e^{-z'/\lambda}\frac{i(0,t-z'/v)}{v}$ | $\frac{e^{-z'/\lambda}}{\lambda}Q(z',t)$ | $\frac{Q(0,t_{RS})}{\lambda}$ |
| BG | - | $\frac{i(0,z'/v)}{v}$ | 0 |
| TCS | $-\frac{i(0,t+z'/c)}{c}$ | $\frac{i(0,z'/v^*)}{v^*}$ | 0 |
| DU | $-\frac{i(0,t+z'/c)}{c} - e^{-(t-z'/v)/\tau_D} \times\left[\frac{i(0,z'/v^*)}{v} + \frac{\tau_D}{v^*}\frac{di(0,z'/v^*)}{dt}\right]$ | $\frac{i(0,z'/v^*)}{v^*} + \frac{\tau_D}{v^*}\frac{di(0,z'/v^*)}{dt}$ | $\frac{\tau_D}{v^*}\frac{di(0,0)}{dt}$ |

TL, transmission line; MTLL, modified transmission line with linear current decay; MTLE, modified transmission line with exponential current decay; BG, Bruce-Golde; TCS, traveling current source; DU, Diendorfer-Uman. In general, the transferred charge density component is non-zero only when the current flows, while the deposited charge density component is non-zero also after the current ceases to flow. The total charge density is the sum of the transferred charge and deposited charge density components. In the simplest return-stroke models, TL and BG, one of the charge density components is absent at all times. In the last column, $\rho_L(0,t_{RS})$ is the total return-stroke charge density at the channel termination on the ground at $t = t_{RS}$ (and afterwards), where $t_{RS}(t_{RS} >> \tau_D)$ is the time when the return-stroke current ceases to flow in all channel sections of interest. For the BG, TCS, and DU models $\rho_L(0,t_{RS})$ is given for the case of $i(0,0) = 0$.

is equal to zero applies to the DU model as well. After a sufficiently long time ($t \to \infty$) the current in the channel ceases to flow and the first three terms in (41), representing the transferred charge, become zero so that

$$\rho_L(z',t) = \frac{1}{v^*}\left[i(0,z'/v^*) + \tau_D \frac{di(0,z'/v^*)}{dt}\right] \qquad (42)$$

Equation (42) represents the charge deposited on the channel by the return-stroke process and is identical to (A13b) of *Diendorfer and Uman* [1990], except for the opposite sign, the disparity being due to different sign conventions for current. The charge density distribution for the DU model at $t$=100 μs obtained using (41), $\tau_D$ = 0.1 μs, and the channel-base current adopted by *Nucci et al.* [1990] is presented in Figure 2.

**Validation of return-stroke models using published experimental data.** Expressions for the charge density as a function of height and time for the six return-stroke models are summarized in Table 1 and illustrated in Figures 3a and 3b. Note that the representation of the return-stroke models in terms of charge density is equivalent to the more traditional representation in terms of channel current. For the TL model the total charge density distribution along the channel is the same as that of the traveling current wave (see Figure 3a). Thus, at times greater than the risetime of the wave the TL model is associated with a progressive decrease of charge at the bottom of the channel. As a result, at distances of the order of some kilometers this model allows the reproduction of only the first few tens of microseconds of the characteristic electric field ramp observed in the experimental data to last for a hundred of microseconds or more [e.g., *Uman*, 1987], after which the model-predicted field decreases, as shown by *Nucci et al.* [1990, Figure 12]. In the TL model, as distance decreases, the ramp ends and an "abnormal" decrease of the field starts earlier, after a few microseconds at 50 m (see Figure 4). It appears that the TL model is not a realistic model for calculating lightning electric fields at times greater than some tens of microseconds at distances of the order of some kilometers and after only a few microseconds at distances of the order of tens of meters from the channel.

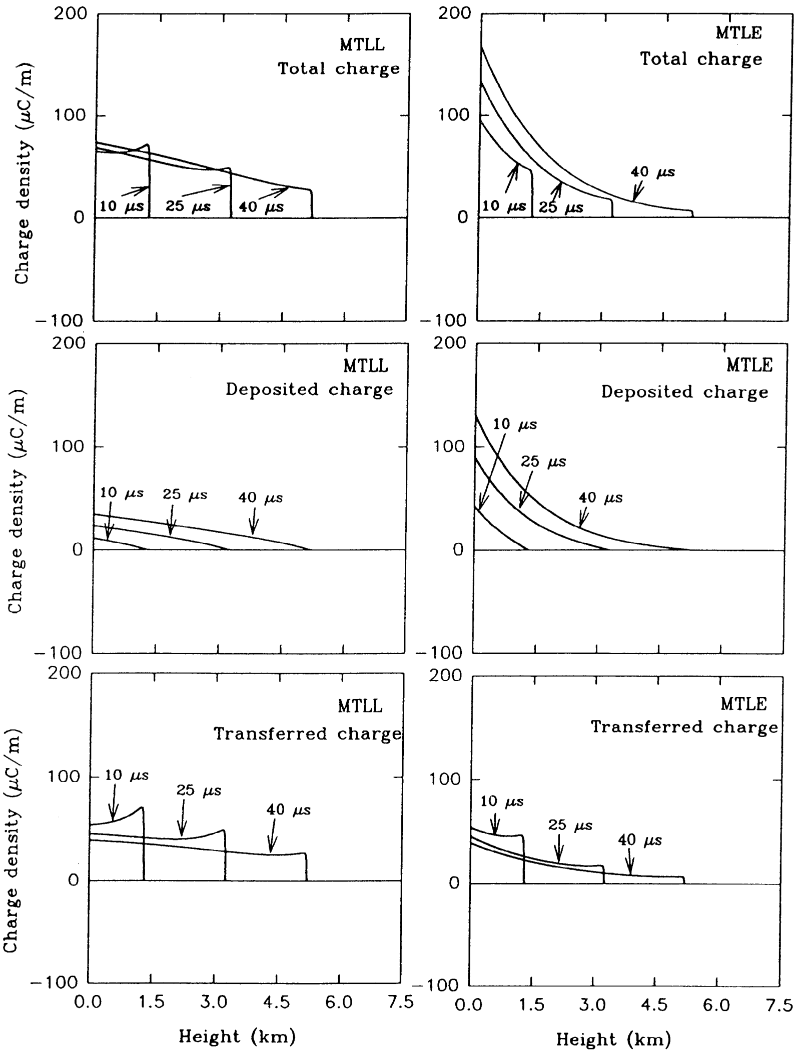


**Figure 3a.** Distributions of the return-stroke charge density (the total charge density and the deposited and transferred charge density components) versus height along the channel at different times for six return-stroke models.

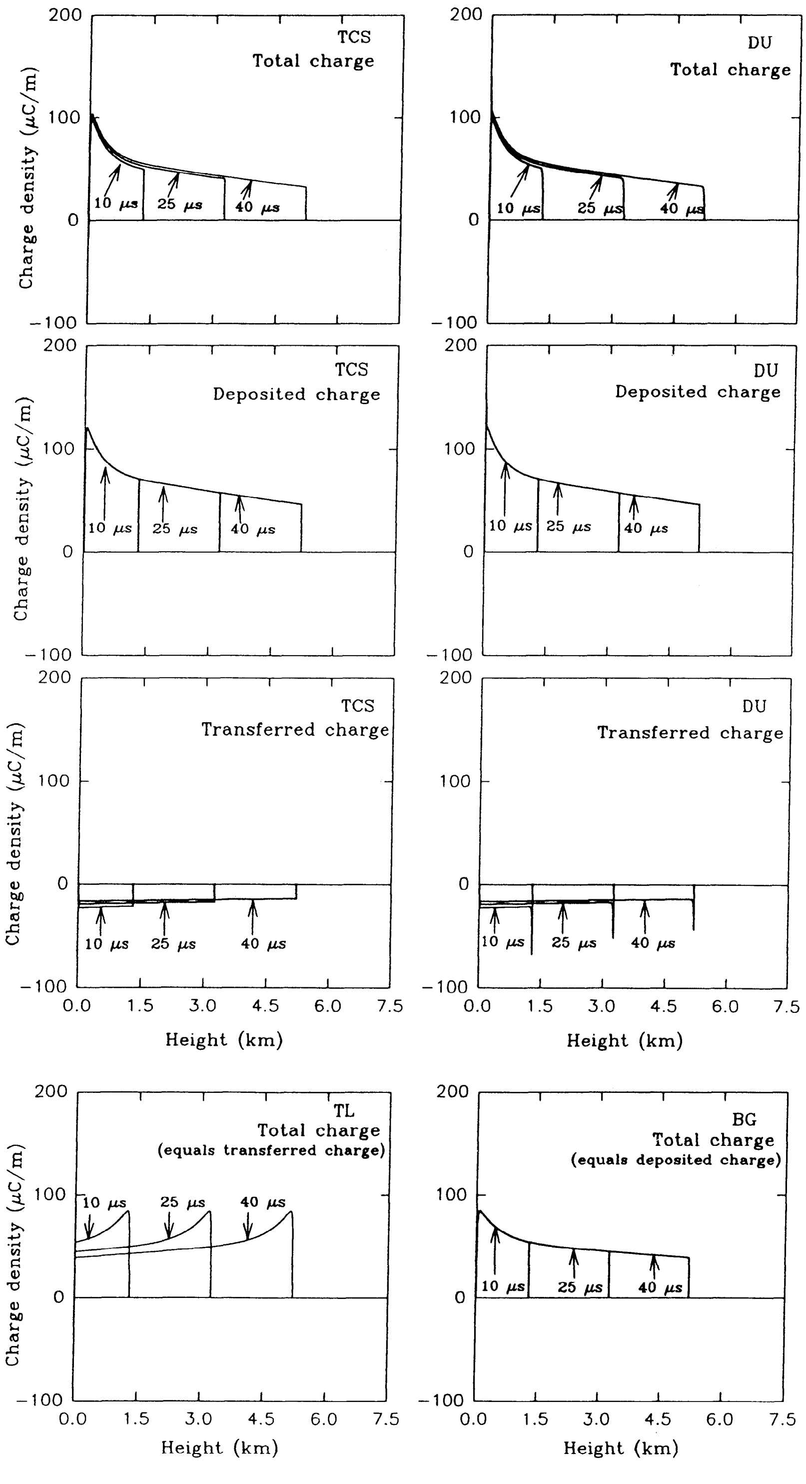


**Figure 3a.** (continued)

At times of the order of 100 μs (see Figure 2), the MTLE model has a total charge density near ground 2 to 3 times higher than that predicted by the MTLL, BG, TCS, and DU models. This disparity translates into an appreciable difference in the model-predicted return-stroke electric field at very close range and in the ratio of the leader-to-return-stroke electric field change at far range, differences which can be used in model validation. Indeed, as seen in Figure 4, at 50 m, the MTLE model predicts a large electric field ramp lasting for many tens of microseconds or longer while for the other models (except for the TL model) the electric field shows little variation after 10 μs or so following the initial relatively fast transition. The measured return-stroke electric fields at 50 m (also at 30 m and 110 m) from triggered lightning, examples of which are shown in Figure 5, adapted from *Uman et al.* [1994], exhibit flattening within the first 10 μs or so, contrary to the prediction of the MTLE model but in support of the other models (except for the TL model) considered here. Note that at distances greater than

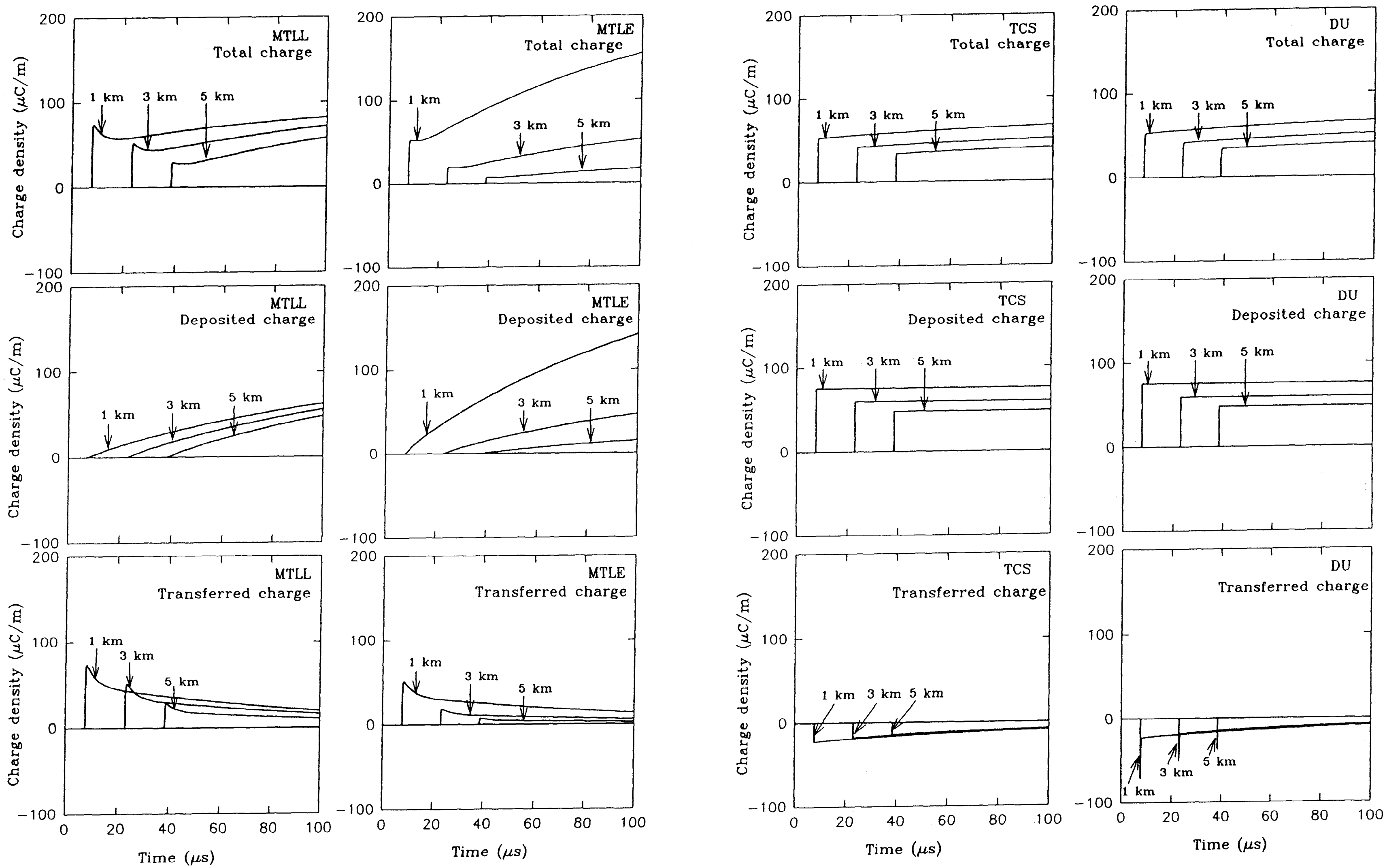


**Figure 3b.** Distributions of the return-stroke charge density (the total charge density and the deposited and transferred charge density components) versus time at different heights on the channel for six return-stroke models.

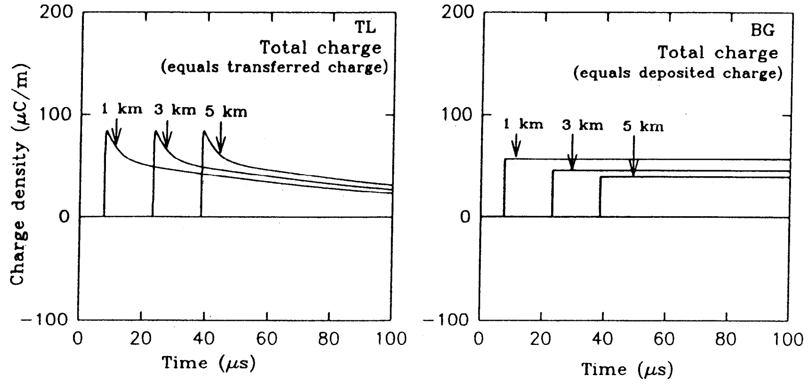


Figure 3b. (continued)

a few kilometers the initial (predominantly radiation) electric field peak is reasonably reproduced by all the return-stroke models considered here, as demonstrated by *Thottappillil and Uman* [1993].

Further, using (8) with $z_t = 0$ and the first term of (4) we calculated the ratio of leader-to-return-stroke electric field as a function of distance as predicted by MTLL, MTLE, BG, TCS, and DU models (see Table 2). For $H = 7.5$ km and for ranges less than 1 km or so, all the models predict essentially the same ratio, whereas at ranges of 20 km or greater there is a considerable difference in the ratio predicted by the MTLE model and all the other models. At 100 km the MTLE ratio is 2 to 3 times greater than that for the MTLL, BG, TCS, and DU models. For $H = 5$ km and $H = 10$ km the ratios predicted by the MTLE model at $r = 100$ km are 2.2 and 4.1, respectively. For the other models the ratio is not sensitive to changes in $H$ from 5 to 10 km. The leader and return-stroke field changes at far ranges have the same polarity. The leader field change can be viewed as being due to moving an equivalent point charge (equal to the leader charge) from its original position in the cloud to some lower position which depends on the leader charge density distribution along the channel, and the return

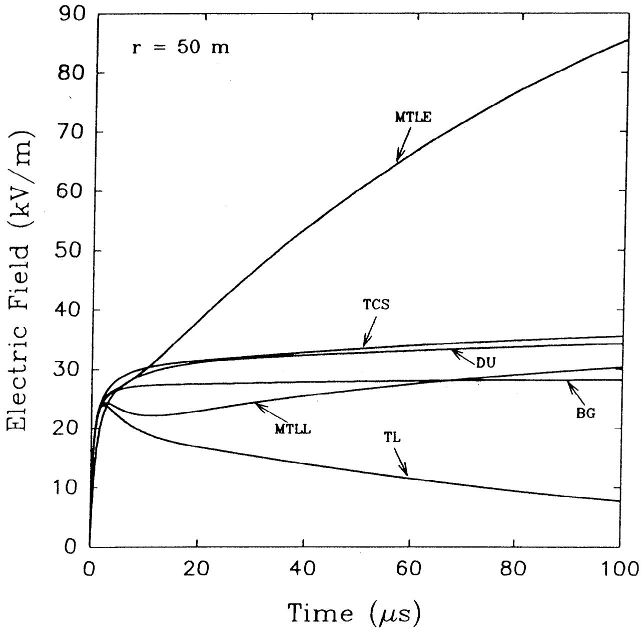


**Figure 4.** Calculated vertical electric fields 50 m from the lightning channel base for six return-stroke models. See also caption of Figure 2.

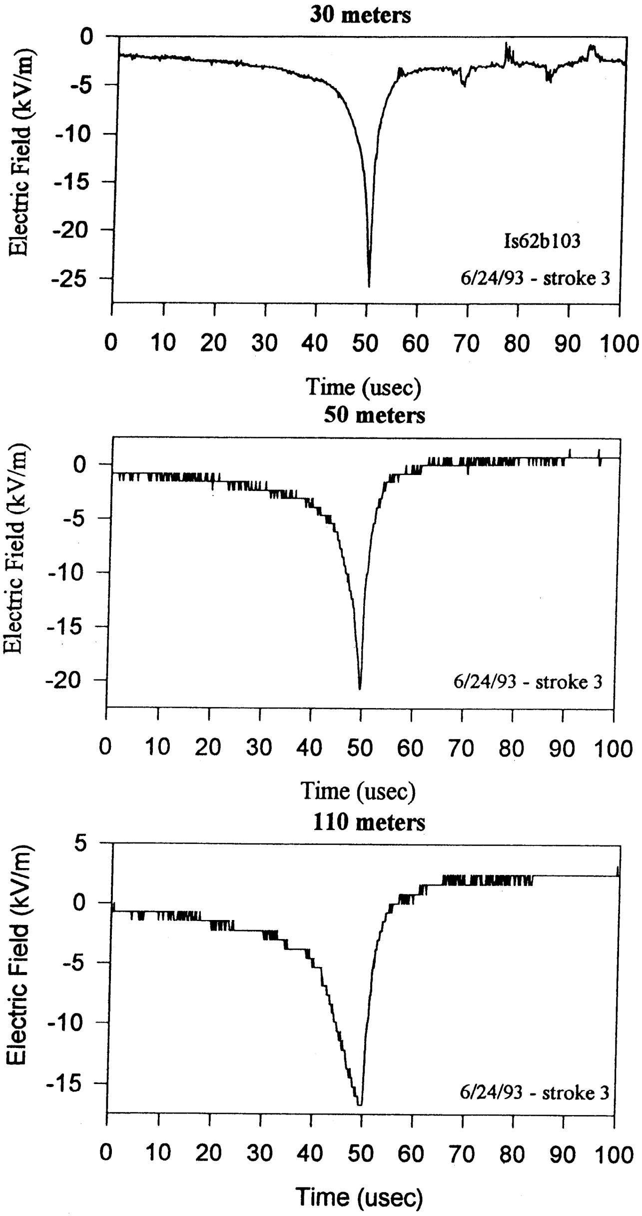


**Figure 5.** Typical measured V-shaped electric field waveforms produced at 30, 50, and 110 m by a leader/return-stroke sequence in triggered lightning. The downward going leading edge of the waveforms is due to the leader while the upward going trailing edge is due to return stroke. The transition from leader to return stroke occurs at the bottom of the V. Note the flattening of the return-stroke field within 10 μs or so. The waveforms are adapted from *Uman et al.* [1994].

**Table 2.** Ratio of Leader-to-Return-Stroke Electric Field as a Function of Distance as Predicted by Five Return-Stroke Models Versus Observations

| Return-Stroke Model | Distance, km | | | | | |
|---|---|---|---|---|---|---|
| | 0.05 | 1 | 5 | 20 | 50 | 100 |
| MTLL | -0.99 | -0.85 | - 0.14 | + 0.81 | + 0.97 | + 0.99 |
| MTLE | -1.0 | -0.92 | + 0.14 | + 2.6 | + 3.0 | + 3.1 |
| BG | -1.0 | -0.87 | - 0.09 | + 1.1 | + 1.2 | + 1.3 |
| TCS | -1.0 | -0.88 | - 0.08 | + 1.1 | + 1.3 | + 1.4 |
| DU | -1.0 | -0.88 | - 0.08 | + 1.1 | + 1.3 | + 1.4 |
| Experimental Data (Mean Values) | -1.0(6)[a] | -0.81(6)[b] | - 0.17(12)[c] | + 0.8(97)[d] | | - |

Only the deposited charge density component (see Table 1) is used. For the MTLL and MTLE models the deposited charge density component is calculated at $t$ = 1 ms. $H$ = 7.5 km; $\lambda$ = 2 km; $\tau_D$ = 0.1 μs; current at the channel base is the same as that adopted by *Nucci et al.* [1990, Figure 4a]. The numbers in the parentheses indicate the sample sizes.

[a]Uman et al. [1994, Table 1]; triggered-lightning strokes.

[b]Beasley et al. [1982, Figure 23b]; distance range from 1 to 2 km; first strokes in natural lightning.

[c]Rakov et al. [1990, Figure 3a]; distance range from 4 to 6 km; first strokes in natural lightning.

[d]Beasley et al. [1982, Figure 23d]; first strokes in natural lightning.

stroke field change as being due to moving that equivalent charge to ground. For the MTLL model this position is midway between the charge source and ground, while for the BG, TCS, and DU models this position is somewhat closer to ground, and for the MTLE model it is approximately 3 times closer to ground (for $H$ = 7.5 km) than to the cloud charge source. Table 2 also gives the experimentally observed leader-to-return-stroke electric field change ratios at various distances taken from the literature [*Beasley et al.*, 1982; *Rakov et al.*, 1990; *Uman et al.*, 1994], which are in support of the MTLL, BG, TCS, and DU models, but not the MTLE model. Thus we conclude that the total charge density distribution for the MTLE model is excessively skewed toward the bottom of the channel causing (1) a large electric field ramp at distances of the order of some tens of meters, not seen in the experimental data (for return strokes in triggered lightning) and (2) too large a leader-to-return-stroke electric field change ratio at far ranges. As seen from Figure 3b, the large electric field ramp predicted by the MTLE model is associated with the deposited charge density component (the second term of (26)). In the MTLL model, as compared to the MTLE model, a larger fraction of the total return-stroke charge is transported to the upper channel sections rather than being deposited on the bottom part of the channel.

Interestingly, from Figure 4, the 50-m electric fields at 100 μs predicted by various models are in approximately the same proportion as the charge density near the bottom of the channel associated with those models (shown in Figure 2, also at $t$ = 100 μs), consistent with (B24). This effect can be readily explained for the MTLL and MTLE models: At later times (strictly speaking when the return-stroke current ceases to flow in all channel sections of interest) the deposited charge density component of $\rho_L(z',t)$ (see Table 1) is dominant; and, further, $Q(z',t) \approx$ const and at very close ranges, such as 50 m, $z' \ll \lambda$ for the portion of the channel "seen" by the observer. As a result, (B24) is applicable and the ratio of the electric field magnitudes for MTLE and MTLL models approaches $H/\lambda$ = 7.5/2 = 3.75, not far from the ratio, about 2.8, of the field magnitudes for the MTLE and MTLL models seen in Figure 4 at 100 μs (when some current still flows in the channel). The field ratio should increase with time (as current approaches zero) getting closer to the value of 3.75 determined by the ratio $H/\lambda$. It appears that (B24) can be a useful tool for estimating the line charge density at the bottom of the lightning channel from measurements of return-stroke electric fields at very close ranges.

## Summary

1. Exact expressions for lightning return-stroke electric and magnetic fields as a function of the charge density along the lightning channel, equivalent to the widely used field expressions in terms of channel current [e.g., *Uman*, 1987], have been derived. The expressions are valid for any return-stroke model. Exact expressions for the leader electric and magnetic fields as a function of both space- and time-varying charge density are also obtained.

2. The charge density distributions along the lightning channel for six return-stroke models have been determined and compared. The charge density distribution for the TL model is the same as that of a traveling current wave and vanishes after the wave has traversed the channel. The MTLE model, in contrast with the MTLL, BG, TCS, and DU models, is associated with a total charge density distribution that is strongly skewed toward the bottom of the channel. As a result, the MTLE model predicts a large electric field ramp at very close ranges not seen in the available experimental data and a ratio of leader-to-return-stroke electric field at far distances that is about 3 times larger than typically observed. Very close electric fields predicted by the MTLL, BG, TCS, and DU models appear to be consistent with the available experimental data for return strokes in triggered lightning. The leader-to-

return-stroke electric field ratio at far ranges for the MTLL, BG, TCS, and DU models is not far from unity, in keeping with the observations. In the TCS and DU models, the charge density distribution during the return-stroke process is influenced by the inherent assumption that the current reflection coefficient at ground is equal to zero. The equations for line charge density given by *Nucci et al.* [1990] for the MTLE and TCS models are incomplete in that they represent only the deposited charge density component rather than the total charge density which is the sum of deposited and transferred charge density components.

### Appendix A: Relation Between Charge Density and Current to be Used for the Formulation of Return-Stroke Models in Terms of Charge Density on the Channel

Consider a section of the current-carrying channel at an arbitrary height $z'$ (Figure 1a). The return-stroke front starts from ground at $t = 0$ and reaches a height $z'$ at $t = z'/v$, where $v$ is the speed of the upward moving front assumed to be a constant. The continuity equation relating the charge per unit length $\rho^*_L(z',t')$ and the current $i^*(z',t')$ in a return-stroke channel at an arbitrary time $t'$ is given by

$$\frac{\partial \rho_L^*(z',t')}{\partial t'} = -\frac{\partial i^*(z',t')}{\partial z'} \tag{A1}$$

where the asterisks are used to denote variables that are specified so that they can be nonzero only at and behind the propagating discharge front, while the same variables without asterisks may be specified as continuous functions which are independent of the position of the moving front on the channel. The return-stroke current and return-stroke charge density do not exist above the return-stroke front and hence can be expressed as

$$i^*(z',t') = u\left(t' - \frac{z'}{v}\right) i(z',t') \tag{A2}$$

$$\rho_L^*(z',t') = u\left(t' - \frac{z'}{v}\right) \rho_L(z',t') \tag{A3}$$

where $u(t' - z'/v)$, the Heaviside function, is defined as

$$\begin{aligned} u(t' - z'/v) &= 0 \quad t' < z'/v \\ u(t' - z'/v) &= 1 \quad t' \geq z'/v \end{aligned} \tag{A4}$$

and $i(z',t')$ and $\rho_L(z',t')$ are continuous functions of $z'$ and $t'$. Note that the continuous functions $i$ and $\rho_L$ can exist above the return-stroke front where there is no actual current or charge density. Substituting (A2) and (A3) in (A1), we get

$$\begin{aligned} &u\left(t' - \frac{z'}{v}\right)\frac{\partial \rho_L(z',t')}{\partial t'} + \delta\left(t' - \frac{z'}{v}\right)\rho_L(z',t') \\ &= -u\left(t' - \frac{z'}{v}\right)\frac{\partial i(z',t')}{\partial z'} + \delta\left(t' - \frac{z'}{v}\right)\frac{1}{v} i(z',t') \end{aligned} \tag{A5a}$$

where $\delta(t' - z'/v)$ is the Dirac delta function which is equal to zero for all values of $t'$ except for $t' = z'/v$, and whose integral over time, including $t' = z'/v$, is equal to unity.

Below the front, $t' > z'/v$, the Heaviside function in (A5a) is equal to unity and the delta function is equal to zero, so that the continuity equation (A5a) reduces to

$$\frac{\partial \rho_L(z',t')}{\partial t'} = -\frac{\partial i(z',t')}{\partial z'} \tag{A5b}$$

Taking the time integral from $-\infty$ to $t' > z'/v$ on both sides of (A5a) and noting that the integral from $-\infty$ to $z'/v$ is zero because the return-stroke front has not yet arrived at the channel section under consideration, we can write

$$\begin{aligned} &\int_{z'/v}^{t'} \frac{\partial \rho_L(z',\tau)}{\partial \tau} d\tau + \rho_L\left(z', \frac{z'}{v}\right) \\ &= -\int_{z'/v}^{t'} \frac{\partial i(z',\tau)}{\partial z'} d\tau + \frac{i(z',z'/v)}{v} \end{aligned} \tag{A6a}$$

At the front, $t' = z'/v$, the integral terms in (A6a) are equal to zero, and the continuity equation becomes

$$\rho_L\left(z', \frac{z'}{v}\right) = \frac{i(z',z'/v)}{v} \tag{A6b}$$

Since $z'$ and $t'$ are independent variables, the first term on the left-hand side of (A6a) can be rearranged as

$$\begin{aligned} \int_{z'/v}^{t'} \frac{\partial \rho_L(z',\tau)}{\partial \tau} d\tau &= \int_{z'/v}^{t'} d\rho_L(z',\tau) \\ &= \rho_L(z',t') - \rho_L(z',z'/v) \end{aligned} \tag{A7}$$

Substituting (A7) in (A6a), we can write,

$$\rho_L(z',t') = \frac{i(z',z'/v)}{v} - \int_{z'/v}^{t'} \frac{\partial i(z',\tau)}{\partial z'} d\tau \tag{A8}$$

Equation (A8) with $t'$ replaced by $t$ is used in the paper to derive the charge density distribution along the lightning channel for six return-stroke models.

## Appendix B

### B1. Relations Between Charge Density and Current to be Used in the Derivation of Field Equations in Terms of Charge Density on the Channel

If a known charge density is to be used in the calculation of the remote electric and magnetic fields, retardation effects must be taken into account. That is, the charge density at an appropriate earlier time, $t - R/c$, contributes to field at time $t$ a distance $R$ away. The following derivation will be given for the case of return strokes with modifications needed for the case of leaders being discussed at the end of this section. Taking into account the retardation effects can be accomplished by replacing $t'$ by $t - R(z')/c$ in (A2) and (A3), where

$$R(z') = \sqrt{z'^2 + r^2}.$$

Then the substitution of (A2) and (A3) into (A1), noting that $dt'/dt = 1$, gives

$$\begin{aligned} &u\left(t - \frac{R(z')}{c} - \frac{z'}{v}\right)\frac{\partial \rho_L(z', t - R(z')/c)}{\partial t} \\ &+ \delta\left(t - \frac{R(z')}{c} - \frac{z'}{v}\right)\rho_L\left(z', t - \frac{R(z')}{c}\right) \\ &= -u\left(t - \frac{R(z')}{c} - \frac{z'}{v}\right)\frac{\partial i(z', t - R(z')/c)}{\partial z'} \\ &+ \delta\left(t - \frac{R(z')}{c} - \frac{z'}{v}\right)\frac{1}{v_a(z')} i\left(z', t - \frac{R(z')}{c}\right) \end{aligned} \quad \text{(B1)}$$

where

$$\begin{aligned} \frac{1}{v_a(z')} &= -\frac{\partial}{\partial z'}\left(t - \frac{\sqrt{z'^2 + r^2}}{c} - \frac{z'}{v}\right) \\ &= \frac{1}{v}\left(1 + \frac{v}{c}\frac{z'}{\sqrt{z'^2 + r^2}}\right) = \frac{1}{v}\left[1 - \frac{v}{c}\cos\theta(z')\right] \end{aligned} \quad \text{(B2)}$$

Angle $\theta(z')$ is defined in Figure 1a, and $v_a(z')$ is the speed of the return-stroke front as "seen" by the observer at $P$ when the front passes $z'$ at time $t'' = R(z')/c + z'/v$. The latter equation differentiated with respect to $t''$ yields $v_a(z') = dz'/dt''$. Equation (B1) is the continuity equation as "seen" by the observer at $P$ at time $t$. It represents an event that occurred a time $R(z')/c$ earlier but was not "seen" until $t$ because of propagation delay. For $t > R(z')/c + z'/v$ (behind the front) the Heaviside function in (B1) is equal to unity and the delta function is equal to zero so that we can write

$$\frac{\partial \rho_L[z', t - R(z')/c]}{\partial t} = -\frac{\partial i[z', t - R(z')/c]}{\partial z'} \quad \text{(B3)}$$

Integrating (B1) from $z'/v + R(z')/c$ to $t$ leads to

$$\begin{aligned} \rho_L\left(z', t - \frac{R(z')}{c}\right) &= \frac{i(z', z'/v)}{v_a(z')} \\ &- \int_{\frac{z'}{v} + \frac{R(z')}{c}}^{t} \frac{\partial i[z', \tau - R(z')/c]}{\partial z'} d\tau \end{aligned} \quad \text{(B4)}$$

Equation (B4) defines the charge density distribution along the channel at any given time $t$ as "seen" by a stationary observer at $P$ at a distance $r$ from the base of the channel (see Figure 1a).

If "$a$" and "$b$" are two independent variables, the Leibnitz's formula for the derivative with respect to $a$ of an integral of the function $f(a,b)$ with limits being functions of $a$, is given by

$$\begin{aligned} \frac{d}{da}\int_{g(a)}^{h(a)} f(a,b)\,db &= \int_{g(a)}^{h(a)} \frac{\partial f(a,b)}{\partial a} db \\ &+ f[a, h(a)]\frac{dh(a)}{da} - f[a, g(a)]\frac{dg(a)}{da} \end{aligned} \quad \text{(B5)}$$

Applying (B5) to (B4) and noting that $z'$ and $t$ are independent, we can rewrite (B4) as

$$\rho_L\left(z', t - \frac{R(z')}{c}\right) = -\frac{d}{dz'}\int_{\frac{z'}{v} + \frac{R(z')}{c}}^{t} i\left(z', \tau - \frac{R(z')}{c}\right) d\tau \quad \text{(B6)}$$

From (B6), we find

$$\begin{aligned} &\rho_L\left(z', t - \frac{R(z')}{c}\right) dz' \\ &= -d\left[\int_{\frac{z'}{v} + \frac{R(z')}{c}}^{t} i\left(z', \tau - \frac{R(z')}{c}\right) d\tau\right] \end{aligned} \quad \text{(B7)}$$

Further, multiplying both sides of (B3) by $dz'$, we find

$$\begin{aligned} &\frac{\partial \rho_L(z', t - R(z')/c)}{\partial t} dz' \\ &= -\frac{\partial i[z', t - R(z')/c]}{\partial z'} dz' = -di\left(z', t - \frac{R(z')}{c}\right) \end{aligned} \quad \text{(B8)}$$

Finally, taking the partial derivative with respect to time of (B8), we obtain

$$\frac{\partial^2 \rho_L[z', t - R(z')/c]}{\partial t^2} dz' = -d\left(\frac{\partial i[z', t - R(z')/c]}{\partial t}\right) \quad \text{(B9)}$$

Equations (B8) and (B9) are valid for any lightning process, while (B6) and (B7) are applicable only to the return-stroke like process. For a leader beginning from a height $H_m$ at time $t$=0 and traveling downward at a speed $v$ the lower limit of the integral in (B6) and (B7) will be the sum of the travel times from the source at $H_m$ to $z'$ and from $z'$ to the observer, that is, $(H_m - z')/v + R(z')/c$. In general, the lower limit of the integrals in (B6) and (B7) is the time, $t_b(z')$, at which the channel current is "seen" to begin at $z'$ by the observer at $P$ and with this modification (B6) and (B7) are valid for any lightning process.

### B2. General Expressions for Differential Electric and Magnetic Fields

The remote differential electric field at ground due to a vertical, current-carrying element $dz'$ above a perfectly conducting earth is given by (1). In the following, (1) will be written in terms of charge density using (B7), (B8), and (B9) and some additional equations derived later. Each term of (1) can be represented, omitting $1/(2\pi\epsilon_0)$, as

$$df_1(z') f_2(z', t) = d[f_1(z') f_2(z', t)] - f_1(z')\, df_2(z', t) \quad \text{(B10)}$$

where $f_2(z',t)$ is the current integral, current, or the current derivative and the total differential $df_1(z')$ is $dz'(2z'^2 - r^2)/R^5(z')$, $dz'(2z'^2 - r^2)/cR^4(z')$, or $-dz' r^2/c^2 R(z')$.

Using (B10), the first (electrostatic) term of (1) can be expanded as follows:

$$dE_s(r,t) = \frac{1}{2\pi\epsilon_0} \cdot d\left(\frac{-z'}{R^3(z')}\int_{t_b(z')}^{t} i(z',\tau-\frac{R(z')}{c})d\tau\right) - \frac{1}{2\pi\epsilon_0}\frac{-z'}{R^3(z')} \cdot d\left(\int_{t_b(z')}^{t} i(z',\tau-\frac{R(z')}{c})d\tau\right) \tag{B11}$$

Applying (B7) to the second term of (B11), we get

$$dE_s(r,t) = \frac{1}{2\pi\epsilon_0} \cdot d\left(\frac{-z'}{R^3(z')}\int_{t_b(z')}^{t} i(z',\tau-\frac{R(z')}{c})d\tau\right) - \frac{1}{2\pi\epsilon_0}\frac{z'}{R^3(z')}\rho_L(z',t-\frac{R(z')}{c})dz' \tag{B12}$$

Similarly, the second (induction) term of (1) can be written as

$$dE_i(r,t) = \frac{1}{2\pi\epsilon_0}d\left([-\frac{3}{2}\frac{z'}{c(z'^2+r^2)} + \frac{1}{2}\frac{\tan^{-1}(z'/r)}{cr}]i(z',t-\frac{R(z')}{c})\right) - \frac{1}{2\pi\epsilon_0}[-\frac{3}{2}\frac{z'}{c(z'^2+r^2)} + \frac{1}{2}\frac{\tan^{-1}(z'/r)}{cr}] \cdot d\left(i(z',t-\frac{R(z')}{c})\right) \tag{B13}$$

Applying (B8) to the second term of (B13), we obtain

$$dE_i(r,t) = \frac{1}{2\pi\epsilon_0}d\left([-\frac{3}{2}\frac{z'}{cR^2(z')} + \frac{1}{2}\frac{\tan^{-1}(z'/r)}{cr}]i(z',t-\frac{R(z')}{c})\right) - \frac{1}{2\pi\epsilon_0}[\frac{3}{2}\frac{z'}{cR^2(z')} - \frac{1}{2}\frac{\tan^{-1}(z'/r)}{cr}] \cdot \frac{\partial\rho_L(z',t-R(z')/c)}{\partial t}dz' \tag{B14}$$

Finally, we find that the third (radiation) term of (1) can be written as

$$dE_r(r,t) = \frac{1}{2\pi\epsilon_0} \cdot d\left(\frac{-z'}{c^2R(z')}\frac{\partial i(z',t-R(z')/c)}{\partial t}\right) - \frac{1}{2\pi\epsilon_0}\frac{-z'}{c^2R(z')}d\left(\frac{\partial i(z',\tau-R(z')/c}{\partial t}\right) \tag{B15}$$

Applying (B9) to the second term of (B15), we get

$$dE_r(r,t) = \frac{1}{2\pi\epsilon_0} \cdot d\left(\frac{-z'}{c^2R(z')}\frac{\partial i(z',t-R(z')/c)}{\partial t}\right) - \frac{1}{2\pi\epsilon_0}\frac{z'}{c^2R(z')}\frac{\partial^2\rho_L(z',t-R(z')/c)}{\partial t^2}dz' \tag{B16}$$

Adding (B12), (B14), and (B16), we obtain expression (B17) for differential electric field, equivalent to (1),

$$\begin{aligned} dE_z(r,t) = &-\frac{1}{2\pi\epsilon_0}\frac{z'}{R^3(z')}\rho_L(z',t-\frac{R(z')}{c})dz' \\ &-\frac{1}{2\pi\epsilon_0}\left[\frac{3}{2}\frac{z'}{cR^2(z')} - \frac{1}{2}\frac{\tan^{-1}(z'/r)}{cr}\right] \cdot \frac{\partial\rho_L[z',t-R(z')/c]}{\partial t}dz' \\ &-\frac{1}{2\pi\epsilon_0}\frac{z'}{c^2R(z')}\frac{\partial^2\rho_L[z',t-R(z')/c]}{\partial t^2}dz' \\ &+\frac{1}{2\pi\epsilon_0}d\left(\frac{-z'}{R^3(z')}\int_{t_b(z')}^{t} i(z',\tau-\frac{R(z')}{c})d\tau\right) \\ &+\frac{1}{2\pi\epsilon_0}d\left([-\frac{3}{2}\frac{z'}{cR^2(z')} + \frac{1}{2}\frac{\tan^{-1}(z'/r)}{cr}]i(z',t-\frac{R(z')}{c})\right) \\ &+\frac{1}{2\pi\epsilon_0}d\left(\frac{-z'}{c^2R(z')}\frac{\partial i[z',t-R(z')/c]}{\partial t}\right) \end{aligned} \tag{B17}$$

In a similar manner the differential magnetic field given by (2) can be rewritten as

$$dB_\phi(r,t) = \frac{1}{2\pi\epsilon_0 c^2}\frac{1}{r}\frac{z'}{R(z')}\cdot\frac{\partial\rho_L[z',t-R(z')/c]}{\partial t}dz'$$
$$+\frac{1}{2\pi\epsilon_0 c^2}\frac{1}{c}\tan^{-1}\left(\frac{z'}{r}\right)\frac{\partial^2\rho_L(z',t-R(z')/c)}{\partial t^2}dz' \tag{B18}$$
$$+\frac{1}{2\pi\epsilon_0 c^2}d\left(\frac{1}{r}\frac{z'}{R(z')}i[z',t-R(z')/c]\right)$$
$$+\frac{1}{2\pi\epsilon_0 c^2}d\left(\frac{1}{c}\tan^{-1}\left(\frac{z'}{r}\right)\frac{\partial i[z',t-R(z')/c]}{\partial t}\right)$$

Note that (B17) and (B18) are general and applicable to any lightning process in a vertical channel above a perfectly conducting ground. Some terms of (B17) and (B18) still contain current. In the following, (B17) and (B18) will be integrated for specific lightning processes (return strokes and leaders) with the results being expressed in terms of charge density only.

### B3. Return-Stroke Electric and Magnetic Fields (General Considerations)

The return-stroke front is assumed to propagate upward with a constant speed $v$ starting from ground at time $t$=0. If $H(t)$ is the "radiating" length of the channel "seen" by the observer, then it appears that the total electric field at $P$ at time $t$ should be given by integrating (1) from 0 to $H(t)$,

$$E_z(r,t) = \frac{1}{2\pi\epsilon_0}\int_0^{H(t)}\Big[\frac{2z'^2-r^2}{R^5(z')}\cdot\int_{\frac{z'}{v}+\frac{R(z')}{c}}^{t} i\left(z',\tau-\frac{R(z')}{c}\right)d\tau$$
$$+\frac{2z'^2-r^2}{cR^4(z')}i\left(z',t-\frac{R(z')}{c}\right)$$
$$-\frac{r^2}{c^2R^3(z')}\frac{\partial i(z',t-R(z')/c)}{\partial t}\Big]dz' \tag{B19}$$

However, the current and current derivative in (B19) are implicitly assumed to be continuous. If there is a current discontinuity at the discharge front (as is the case for the BG and TCS models) the contribution to the field due to the current discontinuity has to be found separately and added to (B19) to get the total field. In the presence of current discontinuity at the discharge front we can divide the interval from 0 to $H(t)$ into two parts, one part from 0 to $H^-(t)$ where the current is continuous and the other part from $H^-(t)$ to $H^-(t)+\Delta h$, $\Delta h \to 0$, corresponding to the current discontinuity, to get

$$E_z(r,t) = \int_0^{H(t)} dE_z(r,t)$$
$$= \int_0^{H^-(t)} dE_z(r,t) + \lim_{\Delta h\to 0}\int_{H^-(t)}^{H^-(t)+\Delta h} dE_z(r,t) \tag{B20}$$

### B4. Return Stroke Electric and Magnetic Fields for the Case of no Current Discontinuity at the Moving Front

The first term of (B20), which corresponds to a continuous current distribution along the channel, can be rewritten as

$$\int_0^{H^-(t)} dE_z(r,t) = \int_0^{H^-(t)} dE_s(r,t)$$
$$+\int_0^{H^-(t)} dE_i(r,t) + \int_0^{H^-(t)} dE_r(r,t) \tag{B21}$$

where $dE_s(r,t)$, $dE_i(r,t)$, and $dE_r(r,t)$ are given by (B12), (B14), and (B16), respectively. We find that the first (electrostatic) term of (B21) is

$$E_s(r,t) = \frac{1}{2\pi\epsilon_0}\cdot\left[\frac{-z'}{R^3(z')}\int_{\frac{z'}{v}+\frac{R(z')}{c}}^{t} i\left(z',\tau-\frac{R(z')}{c}\right)d\tau\right]_0^{H^-(t)}$$
$$-\frac{1}{2\pi\epsilon_0}\int_0^{H^-(t)}\frac{z'}{R^3(z')}\rho_L\left(z',t-\frac{R(z')}{c}\right)dz' \tag{B22}$$

Invoking (3), we can show that the first term of (B22) is zero. Therefore we write

$$E_s(r,t) = -\frac{1}{2\pi\epsilon_0}\int_0^{H^-(t)}\frac{z'}{R^3(z')}\rho_L\left(z',t-\frac{R(z')}{c}\right)dz' \tag{B23}$$

Equation (B23) can also be obtained from Coulomb's law by dividing the charged channel into elements small enough to be approximated by point charges and summing the field contributions due to these elements with the retardation effects being taken into account. It can be shown that if (1) $r << H^-(t)$, (2) retardation effects are neglected, and (3) $\rho_L$ does not vary significantly with height within the channel section contributing to field at $r$,

$$E_s(r,t) \approx -\frac{\rho_L(t)}{2\pi\epsilon_0 r} \tag{B24}$$

Equation (B24), which is similar to (4) of *Rubinstein et al.* [1995] derived for a very close uniformly charged and fully developed leader, tells us that the electrostatic field produced by a very close return stroke is approximately proportional to the charge density (assumed to be more or less height-independent) on the bottom part of the channel. Equation (B24) probably explains why the 50-m electric fields at 100 μs for various return-stroke models (see Figure 4) are roughly in the same proportion as the charge density values near ground (see Figure 2), the feature discussed in more detail in section Validation of Return-Stroke Models Using Published Experimental Data.

The second (induction) term of (B21) can be similarly expressed as

$$E_i(r,t) = \frac{1}{2\pi\epsilon_0} \cdot \left[\left(-\frac{3}{2}\frac{z'}{cR^2(z')} + \frac{1}{2}\frac{\tan^{-1}(z'/r)}{cr}\right) i(z',t-\frac{R(z')}{c})\right]_0^{H^-(t)} - \frac{1}{2\pi\epsilon_0}\int_0^{H^-(t)}\left(\frac{3}{2}\frac{z'}{cR^2(z')} - \frac{1}{2}\frac{\tan^{-1}(z'/r)}{cr}\right)\frac{\partial\rho_L[z',t-R(z')/c]}{\partial t}dz' \tag{B25}$$

When limits $H^-(t)$ and 0 are applied (B25) becomes

$$E_i(r,t) = -\frac{1}{2\pi\epsilon_0}\left(\frac{3}{2}\frac{H(t)}{cR^2(H(t))} - \frac{1}{2}\frac{\tan^{-1}[H(t)/r]}{cr}\right) i\left(H^-(t),\frac{H^-(t)}{v}\right) - \frac{1}{2\pi\epsilon_0}\int_0^{H^-(t)}\left(\frac{3}{2}\frac{z'}{cR^2(z')} - \frac{1}{2}\frac{\tan^{-1}(z'/r)}{cr}\right)\frac{\partial\rho_L[z',t-R(z')/c]}{\partial t}dz' \tag{B26}$$

Setting $z' = H^-(t)$ in (B4) and using (3), we get

$$i[H^-(t), H^-(t)/v] = \rho_L[H^-(t), H^-(t)/v]v_a[H^-(t)] \tag{B27}$$

where

$$v_a[H^-(t)] = \frac{dH^-(t)}{dt} = v\cdot\left[1-\frac{v}{c}\cos[\theta(H^-(t))]\right]^{-1} \tag{B28}$$

is the apparent speed of the return-stroke wavefront; that is, the front speed "seen" by the observer at $P$ (see Figure 1a). Note that an analytical solution for $H^-(t)$ can be obtained from (3).

In a similar manner we find that the third (radiation) term of (B21) can be written as

$$E_r(r,t) = -\frac{1}{2\pi\epsilon_0}\frac{H(t)}{c^2(H(t)^2+r^2)^{1/2}} \cdot \frac{\partial i[H^-(t),H^-(t)/v]}{\partial t} + \frac{1}{2\pi\epsilon_0}\int_0^{H^-(t)}\frac{-z'}{c^2(z'^2+r^2)^{1/2}}\frac{\partial^2\rho_L[z',t-R(z')/c]}{\partial t^2}dz' \tag{B29}$$

Differentiating both sides of (B27) with respect to time, we get

$$\frac{\partial i\left(H^-(t),\frac{H^-(t)}{v}\right)}{\partial t} = \frac{\partial}{\partial t}\left[\rho_L\left(H^-(t),\frac{H^-(t)}{v}\right)v_a(H^-(t))\right] \tag{B30}$$

Substituting (B23), (B26), and (B29) in (B21), using (B27), (B28) and (B30) and replacing for simplicity $H^-(t)$ by $H(t)$, we get the expression (equation (4)) for electric field in terms of charge density distribution for the case of a return stroke, if there is no current discontinuity at the moving front. If there is a current discontinuity at the front, $H(t)$ in (4) is to be understood as $H^-(t)$, the position just below the discharge front discontinuity. In this case, $H(t)$ in (B19) is also to be understood as $H^-(t)$. Equation (4) is equivalent to (B19). All the functions in (4) and in (B19) are continuous.

Adopting a procedure similar to that used for obtaining (4), we can show that the azimuthal component of the magnetic field at ground is given by (5).

### B5. Return-Stroke Fields Due to Current Discontinuity at the Moving Front

We now consider the second term of (B20). If we substitute (1) in the second term of (B20), it is readily seen that as $\Delta h\to 0$ the discontinuity does not contribute to the electrostatic and induction field terms, only to the radiation field term. The radiation field term due to current discontinuity at the discharge front is given by

$$\Delta E_z(r,t) = \lim_{\Delta h\to 0}\int_{H^-(t)}^{H^-(t)+\Delta h}\frac{-1}{2\pi\epsilon_0}\cdot\frac{r^2}{c^2R^3(z')}\frac{\partial i[z',t-R(z')/c]}{\partial t}dz' = -\frac{1}{2\pi\epsilon_0}\frac{r^2}{c^2R^3[H(t)]}i(H(t),\frac{H(t)}{v})\frac{dH(t)}{dt} = -\frac{1}{2\pi\epsilon_0}\frac{r^2}{c^2R^3[H(t)]}\rho_L\left(H(t),\frac{H(t)}{v}\right)v_a^2[H(t)] \tag{B31}$$

Note that (3) and (B27) were used to obtain (B31). To find the total electric field from the return stroke with a current discontinuity at the front, (B31) has to be added to either (4) or (B19).

Analogously, the total magnetic field is obtained by adding to (5) the following term associated with current discontinuity at the front:

$$\Delta B_\phi(r,t) = \frac{1}{2\pi\epsilon_0 c^2}\frac{r}{c\,R^2[H(t)]}\cdot i\left(H(t),\frac{H(t)}{v}\right)\frac{dH(t)}{dt} = \frac{1}{2\pi\epsilon_0 c^2}\frac{r}{cR^2[H(t)]}\rho_L\left(H(t),\frac{H(t)}{v}\right)v_a^2[H(t)] \tag{B32}$$

The ratio of $\Delta E_z$ to $\Delta B_\phi$ is equal to $-c\{r/R[H(t)]\}$, as expected.

**B6. Leader Electric and Magnetic Fields**

The leader is assumed to propagate vertically downward from a charge center at a height $H_m$. The lower end of the leader $h(t)$ at time $t$, as "seen" by an observer at $P$ (see Figure 1b) is given by the solution of (6). The electric field at ground level is given by

$$E_z(r,t) = -\int_{h(t)}^{H_m} dE_z(r,t) \tag{B33}$$

where $dE_z(r,t)$ is given by (B17) in which $t_b(z') = (H_m - z')/v + R(z')/c$. Integrating (B3) along the leader channel and using the relation

$$i\left(h(t),t-\frac{R[h(t)]}{c}\right) = \rho_L\left(h(t),t-\frac{R[h(t)]}{c}\right)\frac{dh(t)}{dt} \tag{B34}$$

we get

$$i\left(H_m,t-\frac{R(H_m)}{c}\right) = -\frac{d}{dt}\int_{h(t)}^{H_m}\rho_L\left(z',t-\frac{R(z')}{c}\right)dz' \tag{B35}$$

Integrating both sides of (B35) from $R(H_m)/c$ to $t$, we obtain

$$\int_{\frac{R(H_m)}{c}}^{t} i\left(H_m,\tau-\frac{R(H_m)}{c}\right)d\tau = -\int_{h(t)}^{H_m}\rho_L\left(z',t-\frac{R(z')}{c}\right)dz' \tag{B36}$$

Note that $R(H_m)/c$ is the time at which the observer at $P$ first "sees" the leader emerging from the charge center at $H_m$. Differentiating both sides of (B35) with respect to $t$, we get

$$\frac{di\left(H_m,t-\frac{R(H_m)}{c}\right)}{dt} = -\frac{d^2}{dt^2}\int_{h(t)}^{H_m}\rho_L\left(z',t-\frac{R(z')}{c}\right)dz' \tag{B37}$$

Integrating (B17) as indicated in (B33) in a manner similar to that shown in section B4 and additionally using (B35), (B36), and (B37), we get the following expression for the leader electric field:

$$E_z(r,t) = \frac{1}{2\pi\epsilon_0}\int_{h(t)}^{H_m}\frac{z'}{R^3(z')}\rho_L(z',t-\frac{R(z')}{c})dz'$$

$$+\frac{1}{2\pi\epsilon_0}\int_{h(t)}^{H_m}\left(\frac{3}{2}\frac{z'}{cR^2(z')}-\frac{1}{2}\frac{\tan^{-1}(z'/r)}{cr}\right)\cdot\frac{\partial\rho_L[z',t-R(z')/c]}{\partial t}dz'$$

$$+\frac{1}{2\pi\epsilon_0}\int_{h(t)}^{H_m}\frac{z'}{c^2R(z')}\frac{\partial^2\rho_L[z',t-R(z')/c]}{\partial t^2}dz'$$

$$-\frac{1}{2\pi\epsilon_0}\frac{H_m}{R^3(H_m)}\int_{h(t)}^{H_m}\rho_L\left(z',t-\frac{R(z')}{c}\right)dz'$$

$$-\frac{1}{2\pi\epsilon_0}\left(\frac{3}{2}\frac{H_m}{cR^2(H_m)}-\frac{1}{2}\frac{\tan^{-1}(H_m/r)}{cr}\right)\cdot\frac{d}{dt}\int_{h(t)}^{H_m}\rho_L\left(z',t-\frac{R(z')}{c}\right)dz' \tag{B38}$$

$$-\frac{1}{2\pi\epsilon_0}\left(\frac{3}{2}\frac{h(t)}{cR^2(h(t))}-\frac{1}{2}\frac{\tan^{-1}[h(t)/r]}{cr}\right)\cdot\rho_L\left(h(t),\frac{H_m-h(t)}{v}\right)\frac{dh(t)}{dt}$$

$$-\frac{1}{2\pi\epsilon_0}\frac{H_m}{c^2R(H_m)}\frac{d^2}{dt^2}\int_{h(t)}^{H_m}\rho_L\left(z',t-\frac{R(z')}{c}\right)dz'$$

$$-\frac{1}{2\pi\epsilon_0}\frac{h(t)}{c^2R[h(t)]}\frac{\partial}{\partial t}\left[\rho_L\left(h(t),\frac{H_m-h(t)}{v}\right)\frac{dh(t)}{dt}\right]$$

The magnetic field from the leader can be obtained by integrating (B18) between the limits $h(t)$ and $H_m$ and using (B35), (B36), and (B37),

$$B_\phi(r,t) = -\int_{h(t)}^{H_m} dB_\phi(r,t)$$

$$= \frac{-1}{2\pi\epsilon_0c^2}\int_{h(t)}^{H_m}\frac{1}{r}\frac{z'}{R(z')}\frac{\partial\rho_L\left(z',t-\frac{R(z')}{c}\right)}{\partial t}dz'$$

$$-\frac{1}{2\pi\epsilon_0c^2}\int_{h(t)}^{H_m}\frac{1}{c}\tan^{-1}\left(\frac{z'}{r}\right)\frac{\partial^2\rho_L\left(z',t-\frac{R(z')}{c}\right)}{\partial t^2}dz'$$

$$+\frac{1}{2\pi\epsilon_0c^2}\left[\frac{1}{r}\frac{H_m}{R(H_m)}\frac{d}{dt}\int_{h(t)}^{H_m}\rho_L\left(z,t-\frac{R(z')}{c}\right)dz'\right] \tag{B39}$$

$$+\frac{1}{2\pi\epsilon_0c^2}\left[\frac{1}{r}\frac{h(t)}{R(h(t))}\rho_L\left(h(t),t-\frac{R[h(t)]}{c}\right)\frac{dh(t)}{dt}\right]$$

$$+\frac{1}{2\pi\epsilon_0c^2}\left[\frac{1}{c}\tan^{-1}\left(\frac{H_m}{r}\right)\frac{d^2}{dt^2}\int_{h(t)}^{H_m}\rho_L\left(z',t-\frac{R(z')}{c}\right)dz'\right]$$

$$+\frac{1}{2\pi\epsilon_0c^2}\left\{\frac{1}{c}\tan^{-1}\left(\frac{h(t)}{r}\right)\cdot\frac{\partial}{\partial t}\left[\rho_L\left(h(t),t-\frac{R(H_m)}{c}\right)\frac{dh(t)}{dt}\right]\right\}$$

In (B38) and (B39), $dh(t)/dt$ can be obtained by differentiating both sides of (6) with respect to time and rearranging the terms.

**Acknowledgments.** This research was supported in part by NSF grants ATM-9014085 and ATM-9415507 (Program Director R. C. Taylor). The authors would like to thank D. K. McLain, C. A. Nucci, and F. Rachidi for their comments on the paper. Equation (26) was independently derived by C. A. Nucci and F. Rachidi.

## References

Beasley, W.H., M.A. Uman, and P.L. Rustan, Electric fields preceding cloud-to-ground lightning flashes, *J. Geophys. Res.*, *87*, 4883-4902, 1982.

Bruce, C.E.R., and R.H. Golde, The lightning discharge, *J. Inst. Electr. Eng. London*, *88*, 487-520, 1941.

Diendorfer, G., and M.A. Uman, An improved return stroke model with specified channel-base current, *J. Geophys. Res.*, 95, 13,621-13,644, 1990.

Heidler, F., Traveling current source model for LEMP calculation, in *Proceedings of 6th International Symposium on Electromagnetic Compatibility*, pp. 157-162, ETH Zentrum-IKT, Zurich, Switzerland, 1985.

Heidler, F., and C. Hopf, Influence of the lightning channel termination on the lightning current and lightning electromagnetic impulse, paper presented at International Aerospace and Ground Conference on Lightning and Static Electricity, Mannheim, Germany, May 24-27, 1994.

Lin, Y.T., M.A. Uman, and R.B. Standler, Lightning return stroke models, *J. Geophys. Res.*, *85*, 1571-1583, 1980.

Nucci, C. A., C. Mazzetti, F. Rachidi, and M. Ianoz, On lightning return stroke models for LEMP calculations, paper presented at 19th International Conference on Lightning Protection, Graz, Austria, April 1988.

Nucci, C.A., G. Diendorfer, M.A. Uman, F. Rachidi, M. Ianoz, and C. Mazzetti, Lightning return stroke current models with specified channel-base current: A review and comparison, *J. Geophys. Res.*, *95*, 20,395-20,408, 1990.

Rakov, V.A., and A.A. Dulzon, Calculated electromagnetic fields of lightning return strokes (in Russian), *Tekh. Elektrodinam.*, no. 1, 87-89, 1987.

Rakov, V.A., and A.A. Dulzon, A modified transmission line model for lightning return stroke field calculations, in *Proceedings of 9th International Symposium on EMC*, pp. 229-235, ETH Zentrum-IKT, Zurich, Switzerland, 1991.

Rakov, V.A., M.A. Uman, D.M. Jordan, and C.A. Priore III, Ratio of leader to return stroke electric field change for first and subsequent lightning strokes, *J. Geophys. Res.*, *95*, 16,579-16,587, 1990.

Rubinstein, M., F. Rachidi, M.A. Uman, R. Thottappillil, V.A. Rakov, and C.A. Nucci, Characterization of vertical electric fields 500 m and 30 m form triggered lightning, *J. Geophys. Res.*, *100*, 8863-8872, 1995.

Sadiku, M.N.O., *Elements of Electromagnetics*, 821 pp., Sounders College Publishing, Orlando, Fla., 1994.

Schonland, B.F.J., D.B. Hodges, and H. Collens, Progressive lightning, 5, A comparison of photographic and electrical studies of the discharge process, *Proc. R. Soc. London, Ser. A*, *166*, 56-75, 1938.

Thomson, E.M., A theoretical study of electrostatic field wave shapes from lightning leaders, *J. Geophys. Res.*, *90*, 8125-8135, 1985.

Thottappillil, R., and M. A. Uman, Comparison of return stroke models, *J. Geophys. Res.*, *98*, 22,903-22,914, 1993.

Thottappillil, R, D.K. Mclain, M.A. Uman, and G. Diendorfer, Extension of the Diendorfer-Uman lightning return stroke model to the case of a variable upward return stroke speed and a variable downward discharge current speed, *J. Geophys. Res.*, *96*, 17,143-17,150, 1991.

Uman, M.A., The Lightning Discharge, Academic, San Diego, Calif., 1987.

Uman, M.A., and D.K. McLain, Magnetic field of the lightning return stroke, *J. Geophys. Res.*, *74*, 6899-6910, 1969.

Uman, M.A., D.K. Mclain, and E.P. Krider, The electromagnetic radiation from a finite antenna, *Am. J. Phys.*, *43*, 33-38, 1975.

Uman, M. A., et al., Electric fields close to triggered lightning, paper presented at International Symposium on Electromagnetic Compatibility, Univ. of Rome "La Sapienza," Rome, Sept.13-16, 1994.

V.A. Rakov and M.A. Uman, Department of Electrical and Computer Engineering, University of Florida, 216 Larsen Hall, P.O. Box 116200, Gainesville, FL 32611-6200. (e-mail: rakov@admin.ee.ufl.edu)

R. Thottappillil, Institute of High Voltage Research, Uppsala University, Uppsala 75228, Sweden. (e-mail: hvirt@strix.its.uu.se)

(Received March 8, 1996; revised October 22, 1996; accepted October 31, 1996.)