JOURNAL OF GEOPHYSICAL RESEARCH, VOL. 106, NO. D13, PAGES 14,191–14,205, JULY 16, 2001

On different approaches to calculating lightning electric fields

Rajeev Thottappillil
Ångström Laboratory, Uppsala University, Uppsala, Sweden

Vladimir A. Rakov
Department of Electrical and Computer Engineering, University of Florida, Gainesville, Florida

Abstract. Three different approaches to the computation of lightning electric fields are compared. These approaches are the traditional dipole (Lorentz condition) technique and two versions of the monopole (continuity equation) technique. The latter two techniques are based on two different formulations of the continuity equation, one used by *Thottappillil et al.* [1997] and the other by *Thomson* [1999], the difference between the formulations being related to different treatments of retardation effects. The three approaches involve the same expression for the vector potential but different expressions for the scalar potential. It is analytically shown that the three different expressions for the scalar potential are equivalent and satisfy the Lorentz condition. Further, the three approaches yield the same total fields and the same Poynting vectors. However, expressions in the three approaches for the individual electric field components in the time domain, traditionally identified by their distance dependence as electrostatic, induction, and radiation terms, are different, suggesting that explicit distance dependence is not an adequate identifier. It is shown that the so identified individual field components in the electric field equation in terms of charge density derived by *Thottappillil et al.* [1997] are equivalent to the corresponding field components in the traditional equation for electric field in terms of current based on the dipole technique. However, the individual field components in the electric field equation based on *Thomson*'s [1999] approach are not equivalent to their counterparts in the traditional dipole technique equation. Further, in *Thottappillil et al.*'s [1997] technique and in the traditional dipole technique, the gradient of scalar potential contributes to all three electric field components, while in *Thomson*'s [1999] technique it contributes only to the electrostatic and induction components. Calculations of electric fields at different distances from the lightning channel show that the differences between the corresponding field components identified by their distance dependence in different techniques are considerable at close ranges but become negligible at far ranges.

1. Introduction

Rubinstein and Uman [1989] discussed two equivalent approaches to calculating the electric fields produced by a specified source. The first approach, the so-called dipole technique, involves (1) the specification of current density $\bar{J}$, (2) the use of $\bar{J}$ to find the vector potential $\bar{A}$, (3) the use of $\bar{A}$ and the Lorentz condition to find the scalar potential ϕ, (4) the computation of electric field $\bar{E}$ using $\bar{A}$ and ϕ, and (5) the computation of magnetic field $\bar{B}$ using $\bar{A}$. In this technique the source is described only in terms of current density, and the field equations are expressed only in terms of current. The use of the Lorentz condition assures that the current continuity equation, which is not explicitly used in this technique, is satisfied.

Paper number 2001JD900150.
0148-0227/01/2001JD900150$09.00

The second approach, the so-called monopole technique (a somewhat misleading term), involves (1) the specification of current density $\bar{J}$ (or line charge density ρ), (2) the use of $\bar{J}$ (or ρ) and the continuity equation to find ρ (or $\bar{J}$), (3) the use of $\bar{J}$ to find $\bar{A}$ and ρ to find ϕ, (4) the computation of electric field $\bar{E}$ using $\bar{A}$ and ϕ, and (5) the computation of magnetic field $\bar{B}$ using $\bar{A}$. In this technique the source is described in terms of both current density and line charge density, and the field equations are expressed (1) in terms of charge density, or (2) in terms of current, or (3) in terms of both charge density and current. The current continuity equation is used to relate the current density and charge density. There is no need for the explicit use of the Lorentz condition in this technique, although properly specified scalar and vector potentials do satisfy the Lorentz condition.

It appears that the main distinction between the two techniques is whether the Lorentz condition is used to relate the scalar and vector potentials or whether the continuity equation is used to relate the currents and the charges in order to completely describe the fields. Therefore, in the following, we will refer to these techniques as the Lorentz condition technique and the continuity equation technique.

Thottappillil et al. [1997] used the continuity equation to convert the lightning electric and magnetic field equations, based on the Lorentz condition technique and expressed in terms of current, to equivalent equations in terms of line charge density. *Thomson* [1999] challenged these equations alleging that the formulation of the continuity equation used by *Thottappillil et al.* [1997] was incorrect. *Thomson* [1999] considered a different formulation of the continuity equation and showed that it was equivalent to the Lorentz condition. The two formulations of the continuity equation differ in how retardation effects are accounted for, as discussed in section 2. In the following we present an analysis that allows us to compare the formulations of the continuity equation used by *Thottappillil et al.* [1997] and by *Thomson* [1999] and show that they both are consistent with the Lorentz condition, provided that the appropriate expression for scalar potential is used. We also show that the approach used by *Thottappillil et al.* [1997] predicts electric and magnetic fields (expressed in terms of line charge density) that are identical to fields based on the Lorentz condition technique (expressed in terms of current) in which the continuity equation is not employed. Therefore we conclude that *Thomson*'s [1999] criticism is unwarranted.

Further, *Rubinstein and Uman* [1989] and *Safaeinili and Mina* [1991] demonstrated, for a step function wave propagating along a vertical antenna, that field equations based on the Lorentz condition and continuity equation approaches, although equivalent, are very different in their structure. In this paper we extend their studies to an arbitrary traveling wave and show that the individual electric field components traditionally identified by their distance dependence may be different in different approaches.

2. Analysis and Discussion

2.1. Three Equivalent Expressions for Scalar Potential

We compare three expressions for the scalar potential due to a lightning return stroke in a straight vertical channel with the ground plane not taken into account. These three expressions (cases 1, 2, and 3) differ by the method used to express the scalar potential in terms of current. Case 1 is derived from the Lorentz condition and is used in our analysis as the reference. We will show that cases 2 and 3, which correspond to the formulations of the continuity equation used by *Thomson* [1999] and *Thottappillil et al.* [1997], respectively, are equivalent to case 1. Therefore both formulations of the continuity equation that differ in the way in which retardation effects are accounted for are consistent with the Lorentz condition.

2.1.1. Case 1: The Lorentz condition technique. As shown in Appendix A, the scalar potential on a plane perpendicular to the lightning channel and containing the channel base, derived from the vector potential using the Lorentz condition, is

$$\phi(r,t) = -\frac{1}{4\pi\varepsilon_0}\int_0^{L'(t)}\left[\frac{z'}{R^3(z')}\int_{z'/v+R(z')/c}^{t} i\left(z',\tau-\frac{R(z')}{c}\right)d\tau + \frac{z'}{cR^2(z')}i\left(z',t-\frac{R(z')}{c}\right)\right]dz'. \quad (1)$$

As stated above, this is the reference case that does not require the explicit use of the continuity equation. The ground plane is not taken into account in (1). Note that (1), and other equations for ϕ in this paper, give the change in potential due to return stroke, not the absolute potential.

2.1.2. Case 2: The continuity equation technique, *Thomson*'s [1999] formulation. As shown in Appendix B, the scalar potential for this case, which is equivalent to its counterpart in case 1 and therefore is consistent with the Lorentz condition, is

$$\phi(r,t) = \frac{1}{4\pi\varepsilon_0}\frac{Q(t-r/c)}{r} + \frac{1}{4\pi\varepsilon_0}\int_0^{L'(t)}\frac{1}{R(z')}\rho^*\left(z',t-R(z')/c\right)dz', \quad (2a)$$

where

$$Q(t-r/c) = -\int_{r/c}^{t} i(0,\tau-r/c)d\tau \quad (2b)$$

is a stationary point charge located at $z'=0$, and

$$\frac{\partial\rho^*\left(z',t-R(z')/c\right)}{\partial t} = -\left.\frac{\partial i\left(z',t-R(z')/c\right)}{\partial z'}\right|_{t-R(z')/c=const}. \quad (2c)$$

Equation (2c) is the continuity equation used by *Thomson* [1999]. In (2c) the partial differentiation of retarded current with respect to the source coordinate z' is carried out keeping the retarded time constant. That is, the dependence of $R(z')$ on z' is ignored while taking the partial derivative. Equation (2c) can be arrived at from physical reasoning as follows.

Consider a current-carrying channel segment of length $\Delta z'$ whose center (midpoint) M is at a height z' (Figure 1). Let $q^*(z',t^*)$ be the charge contained in the segment at time t^*. Associated with $q^*(z',t^*)$ is a line charge density, which is defined as

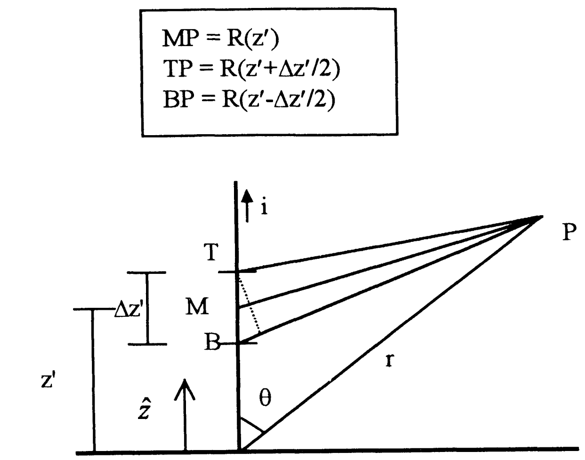

Figure 1. Geometry for explaining the physical meaning of two formulations of the continuity equation, which differ in how retardation effects are accounted for. M is midpoint, T is top point, and B is bottom point. See text for details.

$$\rho^*(z',t^*) = \lim_{\Delta z' \to 0} \frac{q^*(z',t^*)}{\Delta z'}.$$

Charge conservation principle requires that a positive rate of change of charge in segment $\Delta z'$ is equal to a negative net outflow of current from the segment. That is, $\partial q^*(z',t^*)/\partial t^* = -[i(z'+\Delta z'/2,t^*) - i(z'-\Delta z'/2,t^*)]$. Note that currents at the top and bottom boundaries (T and B) of the segment are specified at the same local time t^*. Dividing through by $\Delta z'$ and letting $\Delta z' \to 0$, we can obtain the continuity equation $\partial \rho^*(z',t^*)/\partial t^* = -\partial i(z',t^*)/\partial z'$, with t^* kept constant while carrying out the partial differentiation with respect to z'. The local time t^* could as well be t-$R(z')/c$, where t is the time measured at a remote observation point P at a distance $R(z')$, as shown in Figure 1. Then we can write the continuity equation (2c). *Thottappillil et al.* [1997] used a similar equation to obtain the charge distributions along the channel for different return stroke models (see their equation (A1) and Table 1). However, to convert the electric and magnetic field expressions in terms of current to equivalent expressions in terms of charge density, *Thottappillil et al.* [1997] used a different form of the continuity equation in which retarded time is not kept constant. This situation is treated in case 3 below.

It is easily seen that the point charge $Q(t-r/c)$ at $z'=0$, given by (2b), is required to satisfy the continuity equation at $z'=0$. It is equal in magnitude and opposite in sign to the charge distributed by return stroke along the channel. It represents the amount of charge removed from the origin ($z'=0$) due to the return stroke current flow.

2.1.3. Case 3: The continuity equation technique, *Thottappillil et al.*'s [1997] formulation. As shown in Appendix C, the scalar potential for this case, which is equivalent to its counterpart in case 1 and therefore is consistent with the Lorentz condition, is

$$\begin{aligned}\phi(r,t) &= \frac{1}{4\pi\varepsilon_0}\frac{Q(t-r/c)}{r} \\ &+ \frac{1}{4\pi\varepsilon_0}\int_0^{L'(t)} \frac{\sigma(z',t-R(z')/c)}{R(z')}dz' \\ &+ \frac{1}{4\pi\varepsilon_0}\int_0^{L'(t)} \frac{\rho(z',t-R(z')/c)}{R(z')}dz',\end{aligned} \tag{3a}$$

where

$$Q(t-r/c) = -\int_{r/c}^{t} i(0,\tau - r/c)d\tau,$$

the same as in (2b),

$$\sigma(z',t-R(z')/c)) = -\frac{z'}{cR(z')}i(z',t-R(z')/c) \tag{3b}$$

and

$$\frac{\partial \rho(z',t-R(z')/c)}{\partial t} = -\frac{\partial i(z',t-R(z')/c)}{\partial z'}. \tag{3c}$$

It is possible to derive (3a) starting from (2a). The relationship between $\rho^*(z',t$-$R(z')/c)$ in (2a) and $\rho(z',t$-$R(z')/c)$ in (3a) is derived later in this section. In (3a), σ and ρ are line charge densities and Q is a stationary point charge. Equation (3c) is the continuity equation used by *Thottappillil et al.* [1997] to convert the field equations in terms of current into equivalent equations in terms of charge density. In (3c) the partial differentiation of retarded current with respect to the source coordinate z' is carried out without keeping the retarded time constant; that is, the total-partial or whole-partial derivative [*Brownstein*, 1999] of retarded current is taken.

We now offer, with reference to Figure 1, a physical interpretation of the continuity equation (3c). An observer at P does not "see" the currents at the top (T) and bottom (B) of the segment at the same time. The current at T that observer sees at a given time t is from an earlier time t-$R(z'+\Delta z'/2)/c$, and the current at B that observer sees at time t is from a different earlier time t-$R(z'$-$\Delta z'/2)$. Therefore the rate of change of charge in the channel segment as seen by the observer at P is

$$\frac{\partial q(z',t-R(z')/c)}{\partial t} = -\left\{\begin{matrix} +i(z'+\Delta z'/2, t-R(z'+\Delta z'/2)) \\ -i(z'-\Delta z'/2, t-R(z'-\Delta z'/2)) \end{matrix}\right\}.$$

Dividing through by $\Delta z'$ and letting $\Delta z' \to 0$, we can get (3c), the equation relating charge density and current in the channel as seen by observer at P. The line charge densities corresponding to case 2 (ρ^*) and case 3 (ρ) are different. For the general case of observer above ground, these two charge densities are related by (from equation (B9))

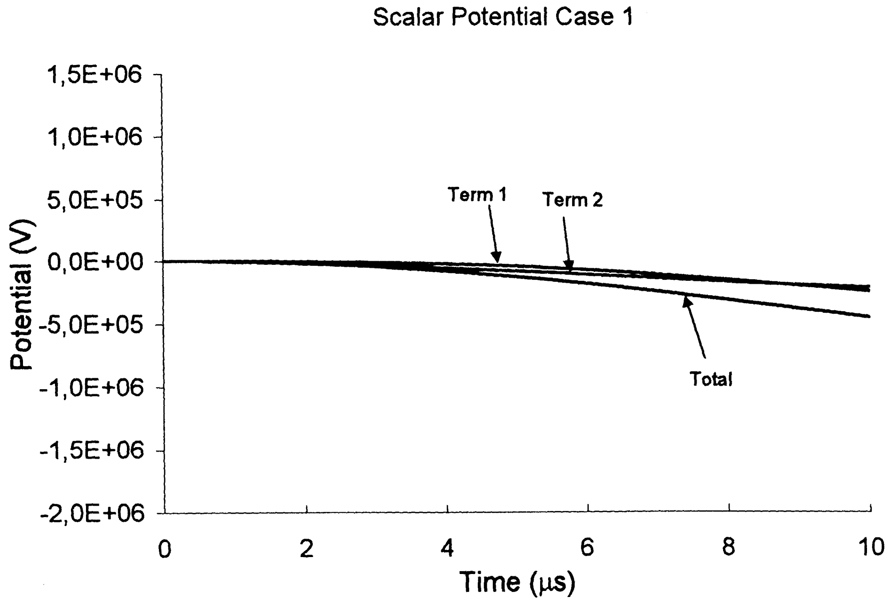

Figure 2. The scalar potential and its components at ground level as predicted by the transmission line model at a distance of 1 km, computed using the dipole (Lorentz condition) technique (equation (1)).

$$\frac{\partial\rho(z',t-R(z')/c)}{\partial t} = \frac{\partial\rho^*(z',t-R(z')/c)}{\partial t} + \frac{z'-r\cos\theta}{cR(z')}\frac{\partial i(z',t-R(z')/c)}{\partial t}, \qquad (3d)$$

where the second term on the right-hand side can be viewed as an adjustment term for the time rate of change of local charge density. Integration of both sides of equation (3d) over time yields

$$\rho(z',t-R(z')/c) = \rho^*(z',t-R(z')/c) + \frac{z'-r\cos\theta}{cR(z')}i(z',t-R(z')/c). \qquad (3e)$$

The factor $(z'-r\cos\theta)/(cR(z')) = -\partial(R/c)/\partial z'$ is the negative rate of change of time retardation with respect to z'.

Although (1), (2a)-(2c), and (3a)-(3c) for the scalar potential are very different in their structure, they are equivalent. Application of (3e) (with $\theta = 90^\circ$) in the scalar potential expressions (3a)-(3c) gives scalar potential expressions (2a)-(2c). Thus (2a)-(2c) and (3a)-(3c) are analytically equivalent. Further, as shown analytically in Appendices B and C, the time derivatives of scalar potentials defined by (1), (2a)-(2c), and (3a)-(3c) are identical (compare equations (B15) and (C7)). Time integration of (B15) or (C7) and changing the order of integration according to (A5) yield (1). The equivalency

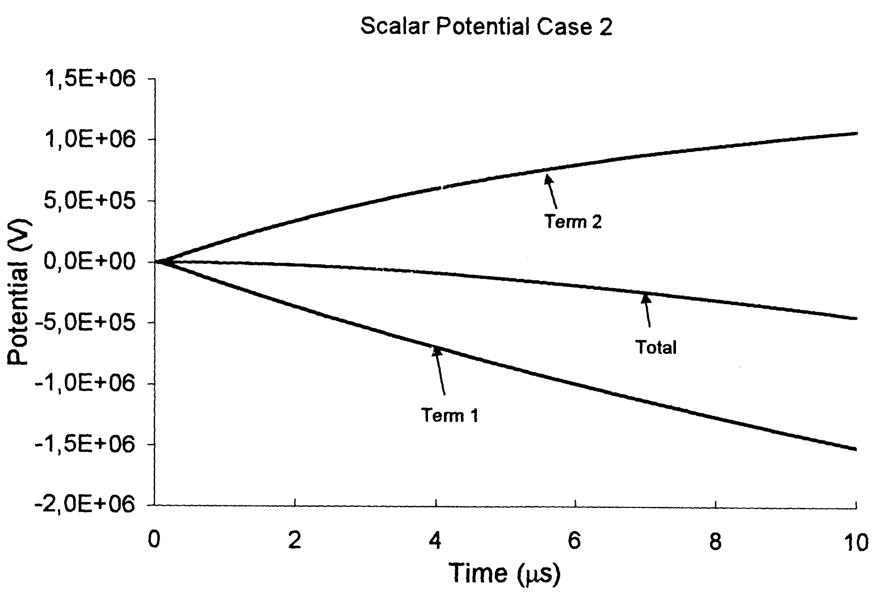

Figure 3. Same as Figure 2, but computed using the continuity equation technique, *Thomson*'s [1999] formulation (equations (2a)-(2c)).

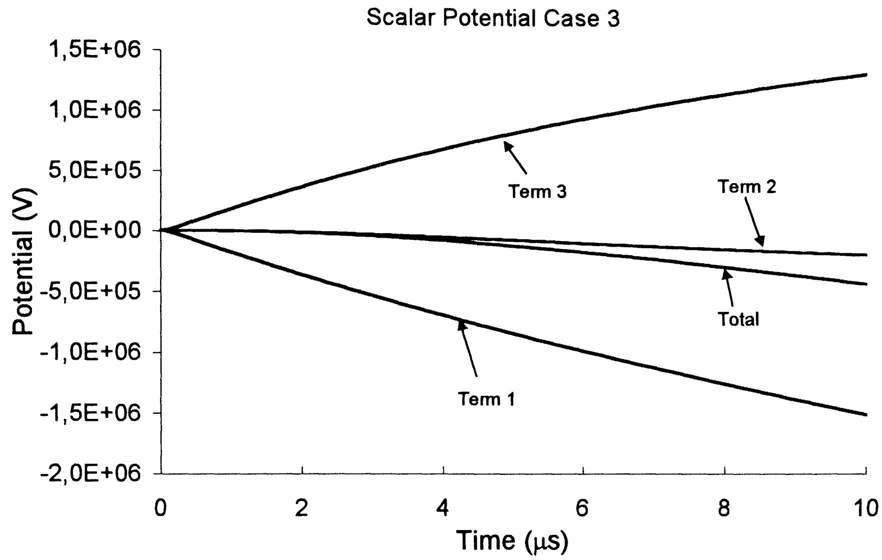

Figure 4. Same as Figure 2, but computed using the continuity equation technique, *Thottappillil et al.*'s [1997] formulation (equations (3a)-(3c)).

is confirmed by calculating numerically the scalar potentials as a function of time at a distance of 1 km using (1), (2a)-(2c), and (3a)-(3c), with the results being shown in Figures 2, 3, and 4, respectively. The transmission line model [e.g., *Rakov and Uman*, 1998] of the return stroke is used in all numerical calculations. As noted above, the ground plane is not taken into account. As seen in Figures 2, 3, and 4, the total potentials for cases 1, 2, and 3, although composed of different components, are identical. It turns out that for the same A , one can specify different, but equivalent expressions for ϕ, that would satisfy the Lorentz condition. In the following we will show that the traditional separation of electric field in the time domain into electrostatic, induction, and radiation components identified by their distance dependence may be different in different techniques, suggesting that explicit distance dependence is not an adequate identifier.

2.2. Individual Field Components

2.2.1. Case 1: The Lorentz condition technique. The traditional expressions for electric and magnetic fields

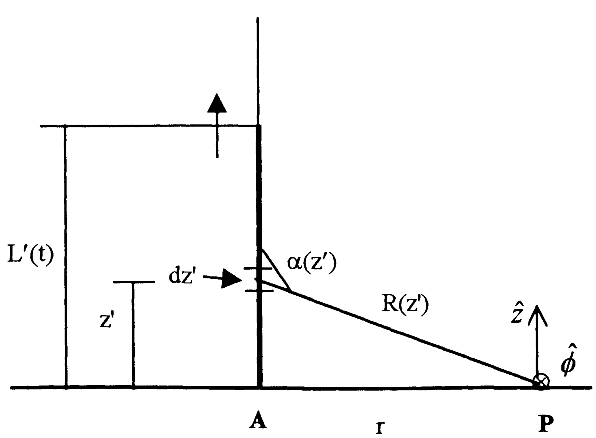

Figure 5. Geometry used in deriving field equations (4), (5), (6), and (8).

at ground level using the dipole (Lorentz condition) technique are given elsewhere [*Uman et al.*, 1975; *Thottappillil et al.*, 1998; *Rakov and Uman*, 1998] and reproduced below. The geometry is shown in Figure 5.

$$E_V(r,t) = \frac{1}{2\pi\varepsilon_0}\int_0^{L'(t)}\frac{2-3\sin^2\alpha(z')}{R^3(z')}\int_{t_b}^{t} i\left(z',\tau - R(z')/c\right)d\tau\, dz' + \frac{1}{2\pi\varepsilon_0}\int_0^{L'(t)}\frac{2-3\sin^2\alpha(z')}{cR^2(z')} i(z',t-R(z')/c)dz' - \frac{1}{2\pi\varepsilon_0}\int_0^{L'(t)}\frac{\sin^2\alpha(z')}{c^2R(z')}\frac{\partial i\left(z',t-R(z')/c\right)}{\partial t}dz' - \frac{1}{2\pi\varepsilon_0}\frac{\sin^2\alpha(L')}{c^2R(L')} i\left(L',t-R(L')/c\right)\frac{dL'}{dt} \quad (4)$$

$$B_\phi(r,t) = \frac{1}{2\pi\varepsilon_0 c^2}\int_0^{L'(t)}\left[\frac{\sin\alpha(z')}{R^2(z')} i\left(z',t-R(z')/c\right) + \frac{\sin\alpha(z')}{cR(z')}\frac{\partial i\left(z',t-R(z')/c\right)}{\partial t}\right]dz' + \frac{1}{2\pi\varepsilon_0 c^2}\frac{\sin\alpha(L')}{cR(L')} i\left(L',t-R(L')/c\right)\frac{dL'}{dt} \quad (5)$$

In (4) and (5), E_V is the vertical (z - direction) electric field at ground level, B_ϕ is the horizontal (ϕ - direction) magnetic field at ground level, and $L'(t)$ is the height of the return-stroke wave front as seen from the observation point. The lower limit of the time integral of the first term in (4), t_b, is the time at which the return stroke wave front has reached the height z' for the first time, as seen from the observation point. As opposed to equations for the scalar potential given above, the effect of ground is taken into account, assuming it is perfectly conducting.

Equations (4) and (5) are valid for any return stroke model. Individual terms on the right-hand side of (4) are labeled the electrostatic, induction, and radiation components, and on the right hand side of (5) they are the magnetostatic (or induction) and radiation components. It is customary to identify the electrostatic component by its R^{-3} distance dependence, induction components by their R^{-2} dependence, and radiation components by R^{-1} dependence [e.g., *Thomson*, 1999]. The last term in (4) and the last term in (5) become zero if there is no current discontinuity at the propagating wave front, i.e. if $i(L',t-R(L')/c) = 0$. Far away from the return stroke channel, $z' \ll r$, $\alpha \approx 90^\circ$ and hence $\sin\alpha = 1$. Therefore in (4), with the last term dropped, the factors in front of the integral of current (electrostatic term), current (induction or intermediate term), and time derivative of the current (radiation term) can be approximated as R^{-3}, $c^{-1}R^{-2}$, and $c^{-2}R^{-1}$, respectively. Thus, far away from the channel the electrostatic, induction, and radiation terms for a differential channel segment fall off as R^{-3}, R^{-2}, and R^{-1}, respectively, but at closer distances, $\sin\alpha$ is additionally involved.

2.2.2. Case 2: The continuity equation technique, *Thomson*'s [1999] formulation. As shown in Appendix D, electric and magnetic field equations can also be derived using the continuity equation technique corresponding to case 2 with the expression for vertical electric field at ground level being given by

$$
\begin{aligned}
E_V(r,t) \; = \; & -\frac{1}{2\pi\varepsilon_0}\int_0^{L'(t)} \frac{z'}{R^3(z')}\rho^*\left(z',t-R(z')/c\right)dz' \\
& -\frac{1}{2\pi\varepsilon_0}\int_0^{L'(t)} \frac{z'}{cR^2(z')}\frac{\partial\rho^*\left(z',t-R(z')/c\right)}{\partial t}dz' \\
& -\frac{1}{2\pi\varepsilon_0}\int_0^{L'(t)} \frac{1}{c^2R(z')}\frac{\partial i\left(z',t-R(z')/c\right)}{\partial t}dz' \\
& -\frac{1}{2\pi\varepsilon_0}\frac{L'(t)}{cR^2(L')}\rho^*\left(L',t-R(L')/c\right)\frac{dL'}{dt} \\
& -\frac{1}{2\pi\varepsilon_0}\frac{1}{c^2R(L')}i\left(L',t-R(L')/c\right)\frac{dL'}{dt}.
\end{aligned}
\qquad (6)
$$

The magnetic field expression using this approach is identical to (5), since it is completely determined by the vector potential. Note that by definition, the current and charge density in (6) are related by the continuity equation (2c) used by *Thomson* [1999]. Equation (6) contains both current and charge density, while (4) contains only current. The first three terms of (6) are similar to the corresponding terms of the expression for E-field derived by *Jefimenko* [1989, p. 516] for a charge and current distribution whose boundary is fixed in space. By analogy with (4), the first term of (6) can be considered as representing the electrostatic field (R^{-3} dependence), the sum of second and fourth terms as representing the induction field ($c^{-1}R^{-2}$ dependence), and the sum of third and last terms as representing the radiation field ($c^{-2}R^{-1}$ dependence). It appears that the electrostatic, induction, and radiation terms (except for the last two terms associated with the wave front) in (6) can also be identified as containing z' times line charge density (charge), z' times time derivative of line charge density (time derivative of charge or current), and derivative of current, respectively. If there is no current or charge density discontinuity at the wave front, the last two terms become zero.

It can be shown that while the total fields given by (6) and (4) are identical, the individual field components (electrostatic, induction, and radiation terms identified by their dependence on R) in these two equations are different. This was verified by calculating the individual field components and the total fields using six different return stroke models (Bruce-Golde (BG), Traveling Current Source (TCS), Diendorfer-Uman (DU), Transmission Line (TL), Modified Transmission Line with Linear current decay with height (MTLL), and Modified Transmission Line with Exponential current decay with height (MTLE) models described by *Rakov and Uman* [1998]). Of these six models, the BG and TCS models have current discontinuity at the wave front whereas other models do not have wave front current discontinuity. As an illustration, the individual field components and the total fields at three distances for the TL model are presented here. Since there is no discontinuity at the wavefront for the TL model, the last term of (4) and the last two terms of (6) drop out of the equations. The charge density in (6) is calculated using the continuity equation (2c), which for the transmission line model can be rewritten as [*Thottappillil et al.*, 1997]

$$\rho^*\left(z',t-R(z')/c\right) = \frac{i\left(0,t-z'/v-R(z')/c\right)}{v} \qquad (7)$$

where v is the return stroke speed.

Computed electric fields at distances of 50 m, 1 km, and 100 km are shown in Figures 6, 7, and 8, respectively. In the curve labels in Figures 6, 7, and 8, LC indicates the terms in (4), and CE indicates the terms in (6). The labels EQ, EI, and ER indicate the electrostatic (R^{-3} dependence), induction ($c^{-1}R^{-2}$ dependence), and radiation ($c^{-2}R^{-1}$ dependence) field components. The following can be observed from Figures 6, 7, and 8 and from (6) and (4).

1. The total fields given by (4) and (6) are identical (for up to several decimal places when numbers are compared).

2. In (6) the electrostatic and induction terms are given completely by the gradient of the scalar potential, while the radiation term is completely given by the time derivative of the vector potential. In contrast, in (4), both the gradient of the scalar potential and the time derivative of the vector potential contribute to the radiation field term.

3. The electrostatic (R^{-3} dependence), induction ($c^{-1}R^{-2}$ dependence), and radiation ($c^{-2}R^{-1}$ dependence) terms in (4) are different from the corresponding terms in (6). The

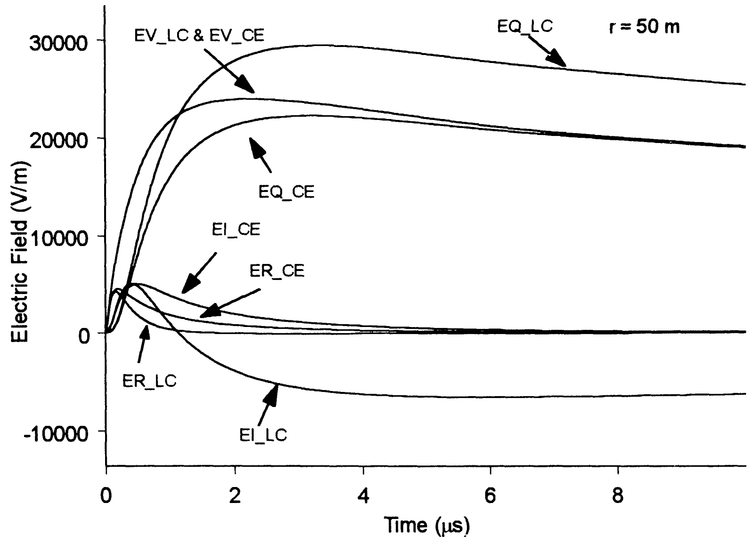

Figure 6. Comparison of the total electric field and its components (identified by their distance dependence) at a distance of 50 m predicted by the transmission line model and field expressions (4) and (6). The labels EQ, EI, and ER indicate the electrostatic (R^{-3} dependence), induction (R^{-2} dependence), and radiation (R^{-1} dependence) field components. LC (Lorentz condition) at the end of the label corresponds to (4), and CE (continuity equation) corresponds to (6).

difference is considerable at 50 m (very close to the channel) and almost negligible at 100 km (far away from the channel).

4. At 50 m (very close to the channel), the electrostatic term (R^{-3} dependence) in (4) is larger than its counterpart in (6) (compare curves EQ_LC and EQ_CE in Figure 6). However, if we assume infinite field propagation speed, the same charge density will be obtained whether we use (2c) or (3c), and the electrostatic terms in (6) and (4) will be equivalent. Thus the difference between the two formulations of the continuity equation is related to different treatments of retardation effects.

The above analysis clearly shows that even though the total electric field from a current or charge distribution is unique, the division of total electric field in the time domain into so-called electrostatic (R^{-3} dependence), induction ($c^{-1}R^{-2}$ dependence), and radiation ($c^{-2}R^{-1}$ dependence) components is not unique. Note that in the Lorentz condition technique, all field components are expressed in terms of current, while in the continuity equation technique, both current and charge density are involved. If the approach described in case 1 is adopted, the gradient of scalar potential contributes to all the three field components, whereas if the approach described in case 2 is adopted, it contributes only to the electrostatic and induction field components. Whether it is case 1 or case 2, the expression for magnetic field at ground level is the same, (5), since it depends only on the vector potential. We get the same Poynting vector whether we calculate it from equation pairs (4) and (5) or (6) and (5),

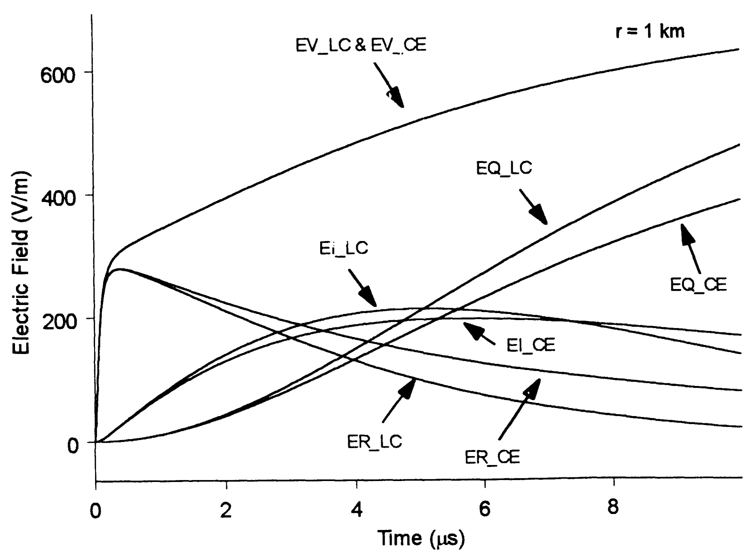

Figure 7. Same as Figure 6, but at a distance of 1 km.

since the total electric fields given by (4) and (6) are the same. In fact, (4) can be analytically derived from (6), as shown in Appendix E.

2.2.3. Case 3: The continuity equation technique, *Thottappillil et al.*'s [1997] formulation. It is possible to start with the scalar potential defined for case 3 (equation 3a) and the corresponding formulation of continuity equation (3c), used by *Thottappillil et al.* [1997], and obtain an expression for electric field using a procedure similar to that adopted in Appendix D. However, field equation corresponding to case 3 is readily obtained by substituting (3d) and (3e), which relate charge densities in case 2 and case 3, into the field expression (6), as shown in Appendix E. The resultant electrical field expression at ground is given below.

$$E_V(r,t) = -\frac{1}{2\pi\varepsilon_0}\int_0^{L'(t)}\frac{z'}{R^3(z')}\rho\left(z',t-R(z')/c\right)dz' + \frac{1}{2\pi\varepsilon_0}\int_0^{L'(t)}\frac{\cos^2\alpha(z')}{cR^2(z')}i\left(z',t-R(z')/c\right)dz' - \frac{1}{2\pi\varepsilon_0}\int_0^{L'(t)}\frac{z'}{cR^2(z')}\frac{\partial\rho\left(z',t-R(z')/c\right)}{\partial t}dz' - \frac{1}{2\pi\varepsilon_0}\int_0^{L'(t)}\frac{\sin^2\alpha(z')}{c^2R(z')}\frac{\partial i\left(z',t-R(z')/c\right)}{\partial t}dz' - \frac{1}{2\pi\varepsilon_0}\frac{L'(t)}{cR^2(L')}\rho\left(L',t-R(L')/c\right)\frac{dL'}{dt} - \frac{1}{2\pi\varepsilon_0}\frac{\sin^2\alpha(L')}{c^2R(L')}i\left(L',t-R(L')/c\right)\frac{dL'}{dt}. \quad (8)$$

Equation (8) is the electric field expression corresponding to case 3, in which the line charge density and current are related by the continuity equation (3c) used by *Thottappillil et al.* [1997]. The first term of (8) is the electrostatic field component (R^{-3} dependence), the sum of second, third, and fifth terms is the induction field component (R^{-2} dependence), and the sum of fourth and last terms is the radiation field component (R^{-1} dependence). It is readily seen that the radiation field component of (4) (the sum of the last two terms) is identical to the radiation field component of (8) (the sum of the fourth and last terms). Equation (8) is analytically equivalent to (6), since the former is derived from the latter. Further, it is shown in Appendix E that (8) is analytically equivalent to the electric field expression derived by *Thottappillil et al.* [1997], the latter expression being reproduced as (9) in section 2.3.

2.3. Expression for Electric Field in Terms of Charge Density of *Thottappillil et al.* [1997]

Thottappillil et al. [1997] started from the electric field expression (4) obtained using the Lorentz condition technique, applied the continuity (3c), and obtained a field expression completely in terms of charge density. The expression for electric field at ground level from *Thottappillil et al.* [1997] is reproduced below (their expressions 4 and B31).

$$E_V(r,t) = -\frac{1}{2\pi\varepsilon_0}\int_0^{L'(t)}\frac{z'}{R^3(z')}\rho(z',t-R(z')/c)dz' - \frac{1}{2\pi\varepsilon_0}\int_0^{L'(t)}\left[\frac{3}{2}\frac{z'}{cR^2(z')} - \frac{1}{2}\frac{\tan^{-1}(\frac{z'}{r})}{cr}\right]\frac{\partial\rho(z',t-R(z')/c)}{\partial t}dz' - \frac{1}{2\pi\varepsilon_0}\int_0^{L'(t)}\frac{z'}{c^2R(z')}\frac{\partial^2\rho(z',t-R(z')/c)}{\partial t^2}dz' - \frac{1}{2\pi\varepsilon_0}\left[\frac{3}{2}\frac{L'(t)}{cR^2(L')} - \frac{1}{2}\frac{\tan^{-1}(\frac{L'}{r})}{cr}\right]\rho(L',t-R(L')/c)\frac{dL'(t)}{dt} - \frac{1}{2\pi\varepsilon_0}\frac{L'(t)}{c^2R(L')}\frac{\partial}{\partial t}\left(\rho(L',t-R(L')/c)\frac{dL'(t)}{dt}\right) - \frac{1}{2\pi\varepsilon_0}\frac{\sin^2\alpha(L')}{c^2R(L')}\rho(L',t-R(L')/c)\left(\frac{dL'(t)}{dt}\right)^2. \quad (9)$$

The current and charge density in (4) and (9), respectively, are related by the continuity equation (3c), used by *Thottappillil et al.* [1997]. The first term of (9) represents the electrostatic (R^{-3} dependence) field component; the sum of second and fourth terms represents the induction (R^{-2} dependence; note that the terms containing the inverse tangent can be transformed using trigonometric functions to obtain the R^{-2} dependence) field component; and the sum of third, fifth, and last terms represents the radiation (R^{-1} dependence) field component. Note that in the last term of (9) $\sin\alpha(L') = r/R(L')$. The last three terms in (9) are nonzero only if there is a discontinuity at the return stroke wave front.

The electrostatic, induction, and radiation components in (9) are equivalent to the corresponding components in (4) because the former are analytically derived from the latter. This is illustrated by a numerical example given below. The vertical electric field on ground at a distance of 1 km was calculated using both (4) and (9). The individual field components and the total field are shown in Figure 9. As before, the transmission line model is used and a constant return stroke speed v is assumed. The charge density in (9) corresponding to the TL model is given by the application of the continuity equation (3c), which can be written in another form as [*Thottappillil et al.*, 1997, equation (B6)]

$$\rho\left(z',t-R(z')/c\right) = -\frac{d}{dz'}\int_{z'/v+R(z')/c}^{t} i\left(z',\tau-R(z')/c\right)d\tau. \quad (10)$$

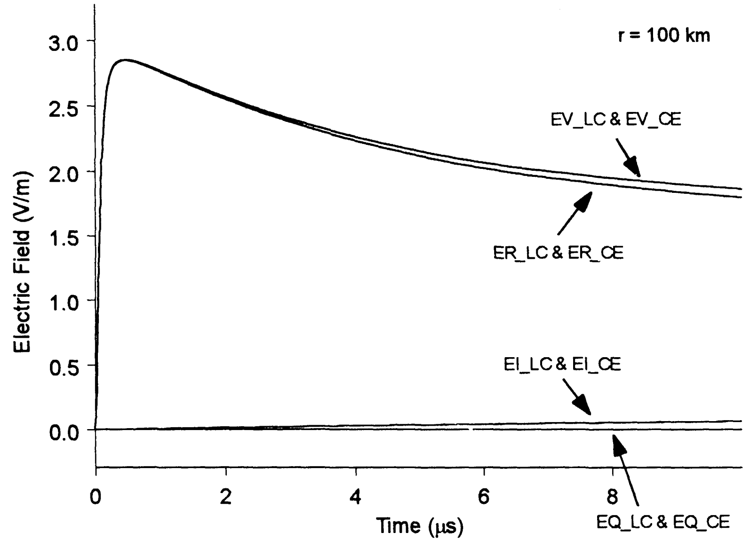

Figure 8. Same as Fig. 6, but at a distance of 100 km.

The spatial and temporal derivatives of retarded currents in the return stroke channel for the TL model are related by [*Thottappillil et al.*, 1998, equation (34)]

$$\frac{\partial i(0,t-z'/v-R(z')/c)}{\partial z'} = -\frac{\partial i(0,t-z'/v-R(z')/c)}{\partial t}\cdot\frac{1}{v}\cdot\left(1+\frac{v}{c}\frac{z'-r\cos\theta}{R(z')}\right). \tag{11}$$

Carrying out the differentiation in (10), applying (11), and noting that for an observer at ground level $\theta = 90°$, we get

$$\rho(z',t-R(z')/c) = \frac{i(0,t-z'/v-R(z')/c)}{v\cdot F_{TL}}, \tag{12}$$

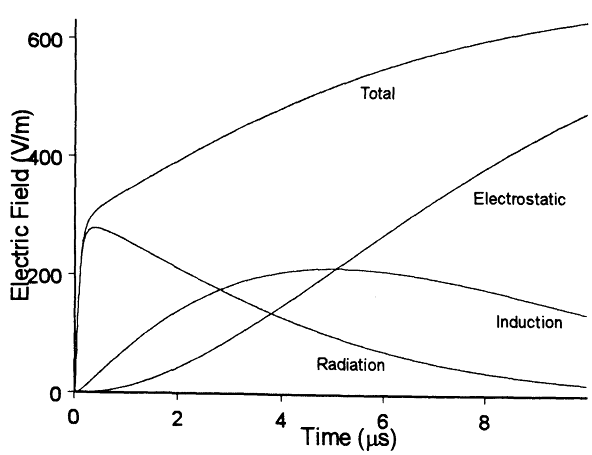

Figure 9. Comparison of the total electric field and the individual field components computed at a distance of 1 km using the transmission line model and field expressions (4) and (9). A total of eight curves are shown, four for (4) and four for (9). Because the agreement between the corresponding curves is excellent, it is impossible to distinguish between them in the plot.

where

$$F_{TL} = \frac{1}{1+v\dfrac{z'}{cR(z')}} . \tag{13}$$

In Figure 9, the total field and individual field components at a distance of 1 km are shown for both (4) and (9). However, the agreement between the two corresponding curves in each case is so good (to several decimal places) that it is impossible to distinguish between them. Equations (4) and (9) were additionally compared by calculating the fields at distances of 50 m and 100 km and again in each case an excellent agreement was found. Further, a comparison of (4) and (9) was also done by calculating the electric fields at 50 m, 1 km, and 100 km, using other return stroke models such as the BG, TCS, DU, MTLL, and MTLE models [*Rakov and Uman*, 1998], and again an excellent agreement was obtained in each case. Thus we conclude that the suggestion of *Thomson* [1999] that the field expressions given by *Thottappillil et al.* [1997] may be erroneous is incorrect.

Appendix A

The objective here is to derive the scalar potential using the vector potential and Lorentz condition (case 1). The lightning return stroke channel can be modeled as a straight line fixed at one end A, with the other end extending with speed v [*Thottappillil et al.,* 1997]. The coordinates are defined in Figure A1. At time $t = 0$ the return stroke starts to propagate from origin A. The observer at the fixed field point P sees the return stroke starting to propagate from the origin at time $t = r/c$, where c is the speed of light. The retarded current at any elemental channel section dz' is given by $i(z',t-R(z')/c)$, where z' is less than or equal to $L'(t)$, the length of the return stroke channel seen by the observer at P at time t. If the return stroke wave front moves at a constant speed v, then $L'(t)$ is obtained from the solution of the equation t

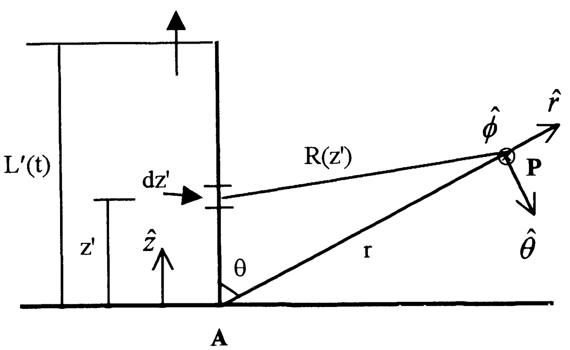

Figure A1. Geometry of the problem.

$= z'/v + R(L')/c$. Note that the assumption of constant return stroke speed is not required in the derivations presented here.

The vector potential at P due to the entire extending channel is given by [*Thottappillil et al.*, 1998, equation (9)]

$$\overline{A}(r,\theta,\tau) = \frac{1}{4\pi\varepsilon_0 c^2} \int_0^{L'(\tau)} \frac{i\left(z', \tau - R(z')/c\right)}{R(z')} \hat{z}\, dz', \tag{A1}$$

where τ is a time less than or equal to time t. At time τ, return stroke wave front is seen at a height $L'(\tau)$ by the observer at P and $L'(\tau)$ is less than or equal to $L'(t)$. Note that in (A1) we have not considered the presence of ground, usually assumed to be perfectly conducting and replaced by the channel image.

The total electric field can be calculated using the relation

$$\overline{E} = -\nabla\phi - \frac{\partial \overline{A}}{\partial t}, \tag{A2}$$

where ϕ can be obtained from the Lorentz condition $\nabla \bullet \overline{A} + (1/c^2)\partial\phi/\partial t = 0$, as

$$\phi(r,\theta,t) = -c^2 \int_{r/c}^{t} \nabla \bullet \overline{A}\, d\tau. \tag{A3}$$

Taking the divergence of (A1), it can be shown that

$$\nabla \bullet \overline{A}(r,\theta,\tau) = \frac{1}{4\pi\varepsilon_0 c^2} \int_0^{L'(\tau)} \left[\frac{z' - r\cos\theta}{R^3(z')} i\left(z', \tau - R(z')/c\right) + \frac{z' - r\cos\theta}{cR^2(z')} \frac{\partial i\left(z', \tau - R(z')/c\right)}{\partial \tau} \right] dz' + \frac{1}{4\pi\varepsilon_0 c^2} \frac{L'(\tau) - r\cos\theta}{cR^2(L')} i\left(L', \tau - R(L')/c\right) \frac{dL'(\tau)}{d\tau}. \tag{A4}$$

Substituting (A4) into (A3) and interchanging the order of integration, an expression for the scalar potential completely in terms of current can be obtained. As time increases from r/c to t, the channel length $L'(\tau)$ increases monotonically from 0 to $L'(t)$. Therefore the order of integration can be changed as follows according to the standard rule:

$$\int_{r/c}^{t} \int_{0}^{L'(\tau)} \Rightarrow \int_{0}^{L'(t)} \int_{\tau}^{t}, \tag{A5}$$

where the lower limit $\tau = t_b$ is the time at which the observer at the field point sees the return-stroke front at height z' for the first time. For a constant return stroke speed v,

$$\tau = \frac{L'(\tau)}{v} + \frac{R(L'(\tau))}{c} = \frac{z'}{v} + \frac{R(z')}{c}.$$

Performing the operations explained above and after some reductions, we get an expression for scalar potential as

$$\phi(r,\theta,t) = -\frac{1}{4\pi\varepsilon_0} \int_0^{L'(t)} \left[\frac{z' - r\cos\theta}{R^3(z')} \int_{\frac{z'}{v} + \frac{R(z')}{c}}^{t} i\left(z', t - R(z')/c\right) d\tau + \frac{z' - r\cos\theta}{cR^2(z')} i\left(z', t - R(z')/c\right) \right] dz'. \tag{A6}$$

From (A6) one can readily obtain expression (A8) of *Uman et al.* [1975], which gives the scalar potential of an elemental current dipole. For a return stroke channel perpendicular to a plane and the observer at this plane ($\theta = 90^\circ$), (A6) reduces to (1).

Appendix B

The objective here is to show that the scalar potential given in case 2 satisfies the Lorentz condition if current and charge density are related by the continuity equation used by *Thomson* [1999]. In Appendix C we will show that there is a different, but equivalent expression for the scalar potential (case 3) that satisfies Lorentz condition if the charge and the current are related by the continuity equation used by *Thottappillil et al.* [1997].

In both Appendices B and C the return stroke channel is assumed to be straight and vertical, and the ground plane is not taken into account. The expression for the potential at P (see Figure A1) corresponding to case 2 is

$$\phi(r,\theta,t) = \frac{1}{4\pi\varepsilon_0} \frac{Q(t - r/c)}{r} + \frac{1}{4\pi\varepsilon_0} \int_0^{L'(t)} \frac{1}{R(z')} \rho^*\left(z', t - R(z')/c\right) dz', \tag{B1}$$

where

$$Q(t - r/c) = -\int_{r/c}^{t} i(0, \tau - r/c)\, d\tau \tag{B2}$$

and

$$\frac{\partial \rho^*(z',t-R(z')/c)}{\partial t} = -\left.\frac{\partial i(z',t-R(z')/c)}{\partial z'}\right|_{t-R(z')/c=const} . \tag{B3}$$

Taking the time derivative of (B1), we obtain

$$\frac{\partial \phi(r,\theta,t)}{\partial t} = -\frac{1}{4\pi\varepsilon_0}\frac{i(0,t-r/c)}{r} + \frac{1}{4\pi\varepsilon_0}\int_0^{L'(t)} \frac{1}{R(z')}\frac{\partial \rho^*(z',t-R(z')/c)}{\partial t}dz' + \frac{1}{4\pi\varepsilon_0}\frac{\rho^*(L',t-R(L')/c)}{R(L')}\frac{dL'}{dt}. \tag{B4}$$

Applying (B3) to the second term of (B4), we get

$$\frac{\partial \phi(r,\theta,t)}{\partial t} = -\frac{1}{4\pi\varepsilon_0}\frac{i(0,t-r/c)}{r} - \frac{1}{4\pi\varepsilon_0}\int_0^{L'(t)} \frac{1}{R(z')}\left.\frac{\partial i(z',t-R(z')/c)}{\partial z'}\right|_{t-\frac{R}{c}=const} dz' + \frac{1}{4\pi\varepsilon_0}\frac{\rho^*(L',t-R(L')/c)}{R(L')}\frac{dL'}{dt}. \tag{B5}$$

Now,

$$\frac{\partial i(z',t-R(z')/c)}{\partial z'} = \left.\frac{\partial i(z',t-R(z')/c)}{\partial z'}\right|_{t-\frac{R}{c}=const} + \frac{\partial i(z',t-R(z')/c)}{\partial (t-R(z')/c)}\frac{\partial (t-R(z')/c)}{\partial z'}, \tag{B6}$$

$$\frac{\partial (t-R(z')/c)}{\partial z'} = -\frac{z'-r\cos\theta}{cR(z')}, \tag{B7}$$

$$\frac{\partial i(z',t-R(z')/c)}{\partial (t-R(z')/c)} = \frac{\partial i(z',t-R(z')/c)}{\partial t}. \tag{B8}$$

Substituting (B8) and (B7) in (B6) and rearranging the terms, we obtain

$$\left.\frac{\partial i(z',t-R(z')/c)}{\partial z'}\right|_{t-R/c=const} = \frac{\partial i(z',t-R(z')/c)}{\partial z'} + \frac{z'-r\cos\theta}{cR(z')}\frac{\partial i(z',t-R(z')/c)}{\partial t}. \tag{B9}$$

Substituting (B9) in (B5), we get

$$\frac{\partial \phi(r,\theta,t)}{\partial t} = -\frac{1}{4\pi\varepsilon_0}\frac{i(0,t-r/c)}{r} - \frac{1}{4\pi\varepsilon_0}\int_0^{L'(t)}\left[\frac{1}{R(z')}\frac{\partial i(z',t-R(z')/c)}{\partial z'} + \frac{z'-r\cos\theta}{cR^2(z')}\frac{\partial i(z',t-R(z')/c)}{\partial t}\right]dz' + \frac{1}{4\pi\varepsilon_0}\frac{\rho^*(L',t-R(L')/c)}{R(L')}\frac{dL'}{dt}. \tag{B10}$$

Integration by parts of the second term of (B10) yields

$$\int_0^{L'(t)} \frac{1}{R(z')}\frac{\partial i(z',t-R(z')/c)}{\partial z'}dz' = \left[\frac{1}{R(z')}i(z',t-R(z')/c)\right]_0^{L'(t)} - \int_0^{L'(t)} \frac{-z'+r\cos\theta}{R^3(z')}i(z',t-R(z')/c)dz', \tag{B11a}$$

which can be simplified as

$$\int_0^{L'(t)} \frac{1}{R(z')}\frac{\partial i(z',t-R(z')/c)}{\partial z'}dz' = \frac{i(L',t-R(L')/c)}{R(L')} - \frac{i(0,t-r/c)}{r} + \int_0^{L'(t)} \frac{z'-r\cos\theta}{R^3(z')}i(z',t-R(z')/c)dz'. \tag{B11b}$$

Substituting (B11b) in (B10) we get

$$\frac{\partial \phi(r,\theta,t)}{\partial t} = -\frac{1}{4\pi\varepsilon_0}\int_0^{L'(t)}\left[\frac{z'-r\cos\theta}{R^3(z')}i(z',t-R(z')/c) + \frac{z'-r\cos\theta}{cR^2(z')}\frac{\partial i(z',t-R(z')/c)}{\partial t}\right]dz' + \frac{1}{4\pi\varepsilon_0}\frac{\rho^*(L',t-R(L')/c)}{R(L')}\frac{dL'(t)}{dt} - \frac{1}{4\pi\varepsilon_0}\frac{i(L',t-R(L')/c)}{R(L')}. \tag{B12}$$

From (3e), for the case $z' = L'(t)$, we get

$$\rho^*(L',t-R(L')/c) = \rho(L',t-R(L')/c) - \frac{L'-r\cos\theta}{cR(L')}i(L',t-R(L')/c). \tag{B13}$$

Also, the relationship between current and charge density at the wave front according to (3c) is given by [*Thottappillil et al.*, 1997] as

$$i(L',t-R(L')/c)=\rho(L',t-R(L')/c)\frac{dL'(t)}{dt}. \tag{B14}$$

Application of (B13) and (B14) in (B12) gives

$$\frac{\partial\phi(r,\theta,t)}{\partial t} = -\frac{1}{4\pi\varepsilon_0}\int_0^{L'(t)}\left[\frac{z'-r\cos\theta}{R^3(z')}i(z',t-R(z')/c)+\frac{z'-r\cos\theta}{cR^2(z')}\frac{\partial i(z',t-R(z')/c)}{\partial t}\right]dz' - \frac{1}{4\pi\varepsilon_0}\frac{L'-r\cos\theta}{cR^2(L')}i(L',t-R(L')/c)\frac{dL'(t)}{dt}. \tag{B15}$$

The divergence of vector potential is derived earlier in Appendix A (see (A4)). From (B15) and (A4) at time t, we can see that the Lorentz condition is satisfied. That is,

$$\nabla\bullet\bar{A}(r,\theta,t)+\frac{1}{c^2}\frac{\partial\phi(r,\theta,t)}{\partial t}=0\ . \tag{B16}$$

Thomson [1999] has also shown that the formulation of continuity equation with retarded time kept constant satisfies the Lorentz condition, but did not present any explicit expression for the scalar potential, such as (B1).

If we write (B15) for a time τ, where τ is less than or equal to t; integrate (B15) between limits r/c to t; and change the order of integration according to (A5), we get an expression for scalar potential identical to that given by (A6) corresponding to case 1.

Appendix C

Here we show that the continuity equation used by *Thottappillil et al.* [1997], i.e., the formulation of the continuity equation without keeping the retarded time constant (the whole-partial derivative of the retarded current with respect to z' is taken), is also consistent with the Lorentz condition, provided that a proper scalar potential is defined. Note that (C1) can be derived from (B1), using (B3), (B9), and (C4).

The assumptions used here are the same as in Appendix B. Let us consider the scalar potential at P (Figure A1),

$$\phi(r,\theta,t) = \frac{1}{4\pi\varepsilon_0}\frac{Q(t-r/c)}{r} + \frac{1}{4\pi\varepsilon_0}\int_0^{L'(t)}\frac{\sigma(z',t-R(z')/c)}{R(z')}dz' + \frac{1}{4\pi\varepsilon_0}\int_0^{L'(t)}\frac{\rho(z',t-R(z')/c)}{R(z')}dz', \tag{C1}$$

where

$$Q(t-r/c)=-\int_{r/c}^{t}i(0,\tau-r/c)d\tau\ , \tag{C2}$$

$$\sigma(z',t-R(z')/c) = -\frac{z'-r\cos\theta}{cR(z')}i(z',t-R(z')/c), \tag{C3}$$

and

$$\frac{\partial\rho(z',t-R(z')/c)}{\partial t}=-\frac{\partial i(z',t-R(z')/c)}{\partial z'}. \tag{C4}$$

Taking the time derivative of (C1), we get

$$\frac{\partial\phi(r,\theta,t)}{\partial t} = -\frac{1}{4\pi\varepsilon_0}\frac{i(0,t-r/c)}{r} - \frac{1}{4\pi\varepsilon_0}\int_0^{L'(t)}\frac{z'-r\cos\theta}{cR^2(z')}\frac{\partial i(z',t-R(z')/c)}{\partial t}dz' + \frac{1}{4\pi\varepsilon_0}\int_0^{L'(t)}\frac{1}{R(z')}\frac{\partial\rho(z',t-R(z')/c)}{\partial t}dz' - \frac{1}{4\pi\varepsilon_0}\frac{L'-r\cos\theta}{cR^2(L')}i(L',t-R(L')/c)\frac{dL'(t)}{dt} + \frac{1}{4\pi\varepsilon_0}\frac{\rho(L',t-R(L')/c)}{R(L')}\frac{dL'(t)}{dt}. \tag{C5}$$

Rewriting the third term of (C5) using (C4) as

$$\frac{1}{4\pi\varepsilon_0}\int_0^{L'(t)}\frac{1}{R(z')}\frac{\partial\rho(z',t-R(z')/c)}{\partial t}dz' = -\frac{1}{4\pi\varepsilon_0}\int_0^{L'(t)}\frac{1}{R(z')}\frac{\partial i(z',t-R(z')/c)}{\partial z'}dz'$$

and integrating by parts, we get

$$\frac{1}{4\pi\varepsilon_0}\int_0^{L'(t)}\frac{1}{R(z')}\frac{\partial\rho(z',t-R(z')/c)}{\partial t}dz' = -\frac{1}{4\pi\varepsilon_0}\frac{i(L',t-R(L')/c))}{R(L')} + \frac{1}{4\pi\varepsilon_0}\frac{i(0,t-r/c)}{r} - \frac{1}{4\pi\varepsilon_0}\int_0^{L'(t)}\frac{z'-r\cos\theta}{R^3(z')}i(z',t-R(z')/c)dz'. \tag{C6}$$

Replacing the third term of (C5) with (C6), applying (B14), we get

$$\frac{\partial\phi(r,\theta,t)}{\partial t} = -\frac{1}{4\pi\varepsilon_0}\int_0^{L'(t)}\left[\frac{z'-r\cos\theta}{R^3(z')}i(z',t-R(z')/c)+\frac{z'-r\cos\theta}{cR^2(z')}\frac{\partial i(z',t-R(z')/c)}{\partial t}\right]dz' - \frac{1}{4\pi\varepsilon_0}\frac{L'-r\cos\theta}{cR^2(L')}i(L',t-R(L')/c)\frac{dL'(t)}{dt}. \tag{C7}$$

Equation (C7) is identical to (B15) and satisfies the Lorentz condition (B16). If we write (C7) for a time τ, where τ is less than or equal to t; integrate (C7) between limits r/c to t; and change the order of integration according to (A5), we get an expression for scalar potential identical to that given by (A6) corresponding to case 1.

Appendix D

The purpose here is to find an expression for electric field corresponding to case 2. Consider the vector and scalar potentials for an extending channel as defined by (A1) and (B1), respectively. The electric field can be obtained using (A2). Using the spherical coordinate system centered at the starting point of the return stroke at ground (Figure A1) and ignoring the presence of ground for the moment, we will show that the negative gradient of the scalar potential $-\nabla\phi$ and the negative time derivative of the vector potential $-\partial\bar{A}/\partial t$ can be found as described below. For $-\nabla\phi$ we have

$$-4\pi\varepsilon_0\nabla\phi = -\hat{r}\frac{\partial}{\partial r}\frac{Q(t-r/c)}{r} - \hat{r}\frac{\partial}{\partial r}\int_0^{L'(t)}\frac{\rho^*(z',t-R(z')/c)}{R(z')}dz' - \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\int_0^{L'(t)}\frac{\rho^*(z',t-R(z')/c)}{R(z')}dz'. \quad \text{(D1)}$$

Note that the first term of (B1) is independent of the spatial coordinate θ. The maximum length of the channel $L'(t)$, as seen from the field point, is a function of r, θ, and t. The distance to the field point from the differential channel segment $R(z')$ is a function of both r and θ, as given by equations (D2a)-(D2c).

$$R(z') = \sqrt{r^2+z'^2-2rz'\cos\theta}, \quad \text{(D2a)}$$

$$\frac{dR(z')}{dr} = \frac{r-z'\cos\theta}{R(z')}, \quad \text{(D2b)}$$

$$\frac{dR(z')}{d\theta} = \frac{rz'\sin\theta}{R(z')}. \quad \text{(D2c)}$$

Carrying out the differentiation of the second and third terms in (D1) and using (D2a)-(D2c), we obtain the following expression:

$$-4\pi\varepsilon_0\nabla\phi = \hat{r}\int_0^{L'(t)}\left[\frac{r-z'\cos\theta}{R^3(z')}\rho^*(z',t-R(z')/c) + \frac{r-z'\cos\theta}{cR^2(z')}\frac{\partial\rho^*(z',t-R(z')/c)}{\partial t}\right]dz'$$
$$+\hat{\theta}\int_0^{L'(t)}\left[\frac{z'\sin\theta}{R^3(z')}\rho^*(z',t-R(z')/c) + \frac{z'\sin\theta}{cR^2(z')}\frac{\partial\rho^*(z',t-R(z')/c)}{\partial t}\right]dz' \quad \text{(D3)}$$
$$-\hat{r}\frac{\rho^*(L',t-R(L')/c)}{R(L')}\frac{\partial L'}{\partial r} - \hat{\theta}\frac{\rho^*(L',t-R(L')/c)}{rR(L')}\frac{\partial L'}{\partial\theta} - \hat{r}\frac{\partial}{\partial r}\frac{Q(t-r/c)}{r}.$$

The time derivative of vector potential is given by

$$-4\pi\varepsilon_0\frac{\partial\bar{A}}{\partial t} = -\hat{z}\int_0^{L'(t)}\frac{1}{c^2R(z')}\frac{\partial i(z',t-R(z')/c)}{\partial t}dz' - \hat{z}\frac{i(L',t-R(L')/c)}{c^2R(L')}\frac{dL'}{dt}, \quad \text{(D4)}$$

where $\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$. (D5)

The general expression for electric field at a field point can be found by combining (D3), (D4), and (D5). We are interested in the return stroke field at ground level. For this case, $\theta = 90°$, and therefore $\cos\theta = 0$, $\sin\theta = 1$, and $\hat{\theta} = -\hat{z}$. The unit vector $\hat{r}$ is now horizontal, pointing away from the channel. A perfectly conducting plane at $z' = 0$ is introduced to simulate the effect of Earth. Using the image theory, we can replace this plane by an image channel carrying current in the same direction as the actual channel. Writing out the equations for image channel and adding them to (D3) and (D4) for the case $\theta = 90°$, we get the expression for electric field, equation (6).

Appendix E

In this appendix the electric field expression corresponding to case 3 is derived. The relationship between the charge densities corresponding to case 2 and case 3 are given by equations (3d) and (3e). For an observer at ground level ($\theta = 90°$), these equations become

$$\frac{\partial\rho(z',t-R(z')/c)}{\partial t} = \frac{\partial\rho^*(z',t-R(z')/c)}{\partial t} + \frac{z'}{cR(z')}\frac{\partial i(z',t-R(z')/c)}{\partial t}, \quad \text{(E1)}$$

$$\rho(z',t-R(z')/c) = \rho^*(z',t-R(z')/c) + \frac{z'}{cR(z')}i(z',t-R(z')/c). \quad \text{(E2)}$$

Substituting (E1) and (E2) in (6), the expression for electric field corresponding to case 2, we get

$$
\begin{aligned}
E_V(r,t) &= \\
&-\frac{1}{2\pi\varepsilon_0}\int_0^{L'(t)} \frac{z'}{R^3(z')}\rho(z',t-R(z')/c)dz' \\
&+\frac{1}{2\pi\varepsilon_0}\int_0^{L'(t)} \frac{z'^2}{cR^4(z')}i(z',t-R(z')/c)dz' \\
&-\frac{1}{2\pi\varepsilon_0}\int_0^{L'(t)} \frac{z'}{cR^2(z')}\frac{\partial\rho(z',t-R(z')/c)}{\partial t}dz' \\
&+\frac{1}{2\pi\varepsilon_0}\int_0^{L'(t)} \frac{z'^2}{c^2R^3(z')}\frac{\partial i(z',t-R(z')/c)}{\partial t}dz' \\
&-\frac{1}{2\pi\varepsilon_0}\int_0^{L'(t)} \frac{1}{c^2R(z')}\frac{\partial i(z',t-R(z')/c)}{\partial t}dz' \\
&-\frac{1}{2\pi\varepsilon_0}\frac{L'(t)}{cR^2(L')}\rho(L',t-R(L')/c)\frac{dL'}{dt} \\
&+\frac{1}{2\pi\varepsilon_0}\frac{L'^2(t)}{c^2R^3(L')}i(L',t-R(L')/c)\frac{dL'}{dt} \\
&-\frac{1}{2\pi\varepsilon_0}\frac{1}{c^2R(L')}i(L',t-R(L')/c)\frac{dL'}{dt}.
\end{aligned}
\qquad \text{(E3)}
$$

After simplification and noting that $\sin\alpha(z') = r/R(z')$, and $\cos\alpha(z') = -z'/R(z')$, (E3) can be written as given by (8). Equation (8) is the electric field expression corresponding to case 3, in which the line charge density and current are related by the continuity equation (3c). The first term (static term) of (8) is identical to the corresponding term in (9), and the fourth and sixth terms (radiation terms) of (8) are identical to the third and fourth terms of (4). In the following it will be shown that the sum of the second, third, and fifth terms (induction terms) of (8) is analytically equivalent to the sum of second and fourth terms (induction terms) of (9).

Integration by parts of the third term of (8) gives

$$
\begin{aligned}
&\frac{1}{2\pi\varepsilon_0}\int_0^{L'(t)} \frac{z'^2}{cR^4(z')}i(z',t-R(z')/c)dz' = \\
&\frac{1}{2\pi\varepsilon_0}i(z',t-R(z')/c)\left[\begin{array}{c} -\frac{1}{2}\frac{z'}{cR^2(z')} \\ +\frac{1}{2}\frac{\tan^{-1}(\frac{z'}{r})}{cr} \end{array}\right]_0^{L'(t)} \\
&-\frac{1}{2\pi\varepsilon_0}\int_0^{L'(t)}\left[\begin{array}{c} -\frac{1}{2}\frac{z'}{cR^2(z')} \\ +\frac{1}{2}\frac{\tan^{-1}(\frac{z'}{r})}{cr} \end{array}\right] di(z',t-R(z')/c).
\end{aligned}
\qquad \text{(E4)}
$$

Using the following two relationships [see *Thottappillil et al.*, 1997]

$$
di(z',t-R(z')/c) = -\frac{\partial\rho(z',t-R(z')/c)}{\partial t}dz', \qquad \text{(E5)}
$$

$$
i(L',t-R(L')/c) = \rho(L',t-R(L')/c)\frac{dL'(t)}{dt}, \qquad \text{(E6)}
$$

equation (E4) can be written as follows:

$$
\begin{aligned}
&\frac{1}{2\pi\varepsilon_0}\int_0^{L'(t)} \frac{z'^2}{cR^4(z')}i(z',t-R(z')/c)dz' = \\
&-\frac{1}{2\pi\varepsilon_0}\left[\begin{array}{c} \frac{1}{2}\frac{L'}{cR^2(L')} \\ -\frac{1}{2}\frac{\tan^{-1}(\frac{L'}{r})}{cr} \end{array}\right]\rho(L',t-R(L')/c)\frac{dL'(t)}{dt} \\
&-\frac{1}{2\pi\varepsilon_0}\int_0^{L'(t)}\left[\begin{array}{c} \frac{1}{2}\frac{z'}{cR^2(z')} \\ -\frac{1}{2}\frac{\tan^{-1}(\frac{z'}{r})}{cr} \end{array}\right]\frac{\partial\rho(z',t-R(z')/c)}{\partial t}dz'.
\end{aligned}
\qquad \text{(E7)}
$$

Substituting (E7) for the second term in (8), applying (E6) to the last term of (8), and adding similar terms, we get

$$
\begin{aligned}
E_V(r,t) &= \\
&-\frac{1}{2\pi\varepsilon_0}\int_0^{L'(t)} \frac{z'}{R^3(z')}\rho(z',t-R(z')/c)dz' \\
&-\frac{1}{2\pi\varepsilon_0}\int_0^{L'(t)}\left[\begin{array}{c} \frac{3}{2}\frac{z'}{cR^2(z')} \\ -\frac{1}{2}\frac{\tan^{-1}(\frac{z'}{r})}{cr} \end{array}\right]\frac{\partial\rho(z',t-R(z')/c)}{\partial t}dz' \\
&-\frac{1}{2\pi\varepsilon_0}\int_0^{L'(t)} \frac{r^2}{c^2R^3(z')}\frac{\partial i(z',t-R(z')/c)}{\partial t}dz' \\
&-\frac{1}{2\pi\varepsilon_0}\left[\begin{array}{c} \frac{3}{2}\frac{L'(t)}{cR^2(L')} \\ -\frac{1}{2}\frac{\tan^{-1}(\frac{L'}{r})}{cr} \end{array}\right]\rho(L',t-R(L')/c)\frac{dL'(t)}{dt} \\
&-\frac{1}{2\pi\varepsilon_0}\frac{r^2}{c^2R^3(L')}\rho(L',t-R(L')/c)\left(\frac{dL'(t)}{dt}\right)^2.
\end{aligned}
\qquad \text{(E8)}
$$

Only the third term in (E8) is in terms of current. This term can be completely expressed in terms of charge density using the continuity equation (3c), as done by *Thottappillil et al.* [1997]. Then expression (E8) becomes identical to expression (9), the electric field expression derived by *Thottappillil et al.* [1997].

Acknowledgments. This research was supported in part by NSF grant ATM 9726100 and by a grant from B. John F. and Svea Andersson. Martin A. Uman, Marcus Berg, and three anonymous reviewers provided a number of useful comments.

References

Brownstein, K. R., The whole-partial derivative, *Am. J. Phys.*, *67*, 639-641, 1999.

Jefimenko, O. D., *Electricity and Magnetism*, 2nd ed., Electret Sci., Star City, West Virginia, 1989.

Rakov, V. A., and M. A. Uman, Review and evaluation of lightning return stroke models including some aspects of their application, *IEEE Trans. Electromagn. Compat.*, *40*, 403-426, 1998.

Rubinstein, M., and M. A. Uman, Methods for calculating the electromagnetic fields from a known source distribution: Application to lightning, *IEEE Trans. Electromagn. Compat.*, *31*, 183-189, 1989.

Safaeinili, A., and M. Mina, On the analytical equivalence of electromagnetic fields solutions from a known source distribution, *IEEE Trans. Electromagn. Compat.*, *33*, 69-71, 1991.

Thomson, E. M., Exact expressions for electric and magnetic fields from a propagating lightning channel with arbitrary orientation, *J. Geophys. Res.*, *104*, 22,293-22,300, 1999.

Thottappillil, R., V. A. Rakov, and M. A. Uman, Distribution of charge along the lightning channel: Relation to remote electric and magnetic fields and to return-stroke models, *J. Geophys. Res.*, *102*, 6987-7006, 1997.

Thottappillil, R., M. A. Uman, and V. A. Rakov, Treatment of retardation effects in calculating the radiated electromagnetic fields from the lightning discharge, *J. Geophys. Res.*, *103*, 9003-9013, 1998.

Uman, M. A., D. K. McLain, and E. P. Krider, The electromagnetic radiation from a finite antenna, *Am. J. Phys.*, *43*, 33-38, 1975.

V. A. Rakov, Department of Electrical and Computer Engineering, University of Florida, 553 Engineering Building # 33, Gainesville, FL 32611. (rakov@ece.ufl.edu)

R. Thottappillil, Ångström Laboratory, Uppsala University, Division for Electricity and Lightning Research, 75121 Uppsala, Sweden. (rajeev.thottappillil@hvi.uu.se)

(Received May 31, 2000; revised February 13, 2001; accepted February 20, 2001.)