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Return Stroke Transmission Line Model for Stroke Speed Near and Equal that of Light

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Abstract. Assuming that the lightning return stroke transmission-line model is applicable, we derive an expression for the return-stroke magnetic field for an arbitrary return stroke speed and from that expression show that for a return stroke speed equal to the speed of light c the electric and magnetic field waveforms at all points in space and the current waveform are identical. Recent measurements indicate that the electric field and current waveforms are similar for about 100 ns for triggered lightning return strokes, potentially implying that the initial return stroke speed is actually near c for that time, that is, for the bottom 30 m or so of the triggered lightning channel. While for the transmission-line model the current waveform and the total electric field waveform are identical for a return stroke speed of c, we show that each of the three individual components (electrostatic, induction, and radiation) that comprise the total field varies significantly with distance.

Introduction

The transmission line model of the lightning return stroke [Uman and McLain, 1969] has been widely used to calculate return stroke currents and current derivatives from measured electric fields and electric field derivatives (e.g., Krider et al., 1996; Weidman and Krider, 1980; Willett et al., 1988, 1989, 1998). In most of these studies, the measured fields and field derivatives were radiation (far) fields, observed many kilometers from the lightning. There have been two recent applications of the transmission line model to close field and field derivatives measured at distances from triggered lightning return strokes (equivalent to subsequent return strokes in natural negative lightning flashes) of 10 to 50 m at which the radiation field is not necessarily dominant (Leteinturier et al., 1990; Uman et al., 2000). In these two cases, the measured electric field and current derivatives were observed to be similar for a time of the order of 100 ns, consistent with a return stroke speed equal the speed of light c, as we will show, although there may be other explanations for this observation. There have also been theoretical arguments advanced that the return stroke speed should be c very early in the return stroke process (Baum, 1990). In the present paper we derive an analytical expression for the magnetic field of the transmission-line model with an arbitrary return stroke speed and a specified channel-base current and from that expression show that for a speed equal c the waveshapes of the electric and magnetic fields at any distance and the current are identical, whereas this is not the case for speeds less than c (e.g., Uman et al., 1975). Further, we discuss the implications of a return stroke value near c, and we provide an example to illustrate that, while for speed c the total electric and magnetic field waveshapes derived from the transmission-line model are invariant with distance and have the same waveshape as the current, each of the three classical electric field components (electrostatic, induction, and radiation) that comprise the total field varies significantly with distance.

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Analysis

The geometry of the problem is illustrated in Figure 1. For the transmission line model 1) the return stroke channel is a vertical line above a perfectly conducting ground plane, and 2) the return stroke current wave propagates upward from ground without distortion or attenuation and without a discontinuity at its wavefront, and is of the form $i(z',t) = i(0,t - z'/v)$. The vector potential $\vec{A}$ at point P is given by,

$$\overline{A}(r,\theta,t) = \frac{1}{4\pi\varepsilon_0 c^2} \int_0^{L'(t)} \frac{i(z',t-R(z'/c)}{R(z')} \hat{z} dz' \qquad (1)$$

with the magnetic flux density found from

$$\overline{B} = \nabla \times \overline{A}. \qquad (2)$$

Equation (2) can be evaluated to obtain,

$$B_\phi(r,t) = \frac{r\sin\theta}{4\pi\varepsilon_0 c^2} \int_0^{L'(t)} \left(\frac{1}{R^3(z')} i(z',t-R(z')/c) + \frac{1}{cR^2(z')} \frac{\partial i(z',t-R(z')/c)}{\partial t} \right) dz'. \qquad (3)$$

Uman et al. (1975) have evaluated Eq. (2) for the case of an observer on the ground plane (that is $\theta = 90^\circ$) and have obtained an expression similar to Eq. (3). In Eq. (1) and Eq. (3), $L'(t)$ is the height of the channel contributing to the field at P at time t [Thottappillil et al., 1997].

For the transmission line model, the "retarded" current is

$$i(z',t-R(z')/c) = i(0,t-z'/v-R(z')/c). \qquad (4)$$

Differentiation of Eq. (4) with respect to z′ yields (see also Eq. (34) of Thottappillil et al., 1998),

$$\frac{\partial i(z',t-R(z')/c)}{\partial z'} = (-\frac{1}{v}-\frac{z'-r\cos\theta}{cR(z')})\frac{\partial i(0,t-z'/v-R(z')/c)}{\partial t}. \tag{5}$$

We now integrate by parts the first term of Eq. (3), using the relation

$$\frac{dz'}{R^3(z')} = \frac{dz'}{(z'^2+r^2-2z'r\cos\theta)^{3/2}} = d(\frac{1}{r^2\sin^2\theta}\frac{z'-r\cos\theta}{R(z')}) \qquad \theta \neq 0. \tag{6}$$

For a non-zero value of θ, the first term of Eq. (3) can be split as

$$= \frac{r\cos\theta}{r^3\sin^2\theta}i(0,t-r/c) + \int_0^{L'(t)}\left(\frac{z'-r\cos\theta}{r^2\sin^2\theta R(z')}(\frac{1}{v}+\frac{z'-r\cos\theta}{cR(z')})\frac{\partial i(0,t-z'/v-R(z')/c)}{\partial t}\right)dz' \tag{7}$$

Substituting Eq. (7) in Eq. (3), we obtain the magnetic flux density in the form,

$$B_\phi(r,t) = \frac{1}{4\pi\varepsilon_0 c^2 r\sin\theta}\int_0^{L'(t)}\left(\frac{z'-r\cos\theta}{vR(z')}+\frac{(z'-r\cos\theta)^2}{cR^2(z')}+\frac{r^2\sin^2\theta}{cR^2(z')}\right) \cdot \frac{\partial i(0,t-z'/v-R(z')/c)}{\partial t}dz' + \frac{\cos\theta}{4\pi\varepsilon_0 c^2 r\sin\theta}i(0,t-r/c). \tag{8}$$

Using mathematical manipulations we can reduce Eq. (8) into the form given in Eq. (9). These manipulations are: Apply $(z'-r\cos\theta)^2+r^2\sin^2\theta = R^2(z')$ to Eq. (8), multiply and divide the resulting expression by $-c/v$, inside the integral add and substract $\frac{1}{v}\frac{\partial i}{\partial t}$, rearrange the terms, apply Eq. (5), and simplify to obtain

$$B_\phi(r,t) = \frac{c/v+\cos\theta}{4\pi\varepsilon_0 c^2 r\sin\theta}i(0,t-r/c) + \frac{1}{4\pi\varepsilon_0 cr\sin\theta}\int_0^{L'(t)}(\frac{1}{c^2}-\frac{1}{v^2})\frac{\partial i(0,t-z'/v-R(z')/c)}{\partial t}dz'. \tag{9}$$

Equation (9) is a general expression for the magnetic field above ground, assuming a transmission line model speed of v, which may or may not be equal to c. Equation (9) is not valid for $\theta = 0$ or π, as noted above. When $\theta = 0$, $B_\phi(r,t) = 0$, which is clearly seen from Eq. (3). The same result is obtained by integrating by parts the first term of Eq. (3) using the relationship

$$\frac{dz'}{R^3(z')} = \frac{dz'}{(z'^2+r^2-2z'r)^{3/2}} = \frac{dz'}{(z'-r)^3} = d(\frac{-1}{2(z'-r)^2}) \tag{10}$$

for the special case $\theta = 0$, applying Eq. (10) to the first term of Eq. (3), and simplifying the resulting expressions.

The presence of ground can be modelled by assuming an image channel, carrying the same current in the same direction, $\hat{z}$. The unit vectors $\hat{r}$ and $\hat{\theta}$ remains unchanged for the image channel. The angle θ appears in the final expressions directly through $\sin\theta$ and $\cos\theta$, and indirectly through $R(z')$ and through the upper limit of integral $L'(t)$. Hence the magnetic field from the image channel is obtained from Eq. (9) by replacing θ by π-θ wherever it occurs. Adding the expression for the magnetic flux density from the image channel to Eq. (9), we obtain the general expression for the magnetic flux density in the presence of an ideal ground.

For the special case $v = c$, Eq. (9) readily reduces to,

$$B_\phi(r,t) = \frac{1+\cos\theta}{4\pi\varepsilon_0 c^2 r\sin\theta}i(0,t-r/c),\ \theta \neq 0\ . \tag{11}$$

Since $\frac{1+\cos\theta}{\sin\theta} = \frac{\sin\theta}{1-\cos\theta}$, Eq. (11) can also be written as

$$B_\phi(r,t) = \frac{\sin\theta}{4\pi\varepsilon_0 c^2 r(1-\cos\theta)}i(0,t-r/c), \theta \neq 0\ . \tag{12}$$

Adding the magnetic field from the image channel to Eq. (12), we find that the total magnetic field at P is

$$B_\phi(r,t)_{Total} = \frac{1}{2\pi\varepsilon_0 c^2 r\sin\theta}i(0,t-r/c), \theta \neq 0\ . \tag{13}$$

The time derivative of Eq. (13) is (The subscript *Total* is dropped henceforth)

$$\frac{\partial B_\phi(r,t)}{\partial t} = \frac{1}{2\pi\varepsilon_0 c^2 r\sin\theta}\frac{\partial i(0,t-r/c)}{\partial t}, \theta \neq 0\ . \tag{14}$$

From Eq. (13) and (14) it is clear that current (time derivative of current) and magnetic field (time derivative of magnetic field) will have the same waveshape at any point in space if $v = c$. Again note that these equations are not valid for $\theta = 0,\pi$. For $\theta = 0$, the current wave arrives at the observer at P at the same instant as the signal from the wavefront (both travel at the speed of light).

At a point on ground, i.e., when $\theta = 90^\circ$,

$$B_\phi(r,t) = \frac{1}{2\pi\varepsilon_0 c^2 r}i(0,t-r/c)\ . \tag{15}$$

In free space the electric field intensity can be obtained from the magnetic flux density by applying Maxwell's equation

$$\nabla\times\overline{B} = \frac{1}{c^2}\frac{\partial\overline{E}}{\partial t} \tag{16}$$

It can be shown that the curl of Eq. (12) is

$$\nabla\times\overline{B} = \frac{1}{2\pi\varepsilon_0 c^3 r\sin\theta}\frac{\partial i(0,t-r/c)}{\partial t}\hat{\theta},\ \theta \neq 0\ . \tag{17}$$

From Eq. (16) and (17) we obtain

$$\frac{\partial\overline{E}}{\partial t} = \frac{1}{2\pi\varepsilon_0 cr\sin\theta}\frac{\partial i(0,t-r/c)}{\partial t}\hat{\theta},\ \theta \neq 0\ . \tag{18}$$

Integrating both sides of Eq. (18) with respect to time between the limits r/c and t, and assuming $i(0,0) = 0$, and $E(r,r/c) = 0$, we find

$$\overline{E}(r,\theta,t) = \frac{1}{2\pi\varepsilon_0 cr\sin\theta}i(0,t-r/c)\hat{\theta}, \theta \neq 0\ . \tag{19}$$

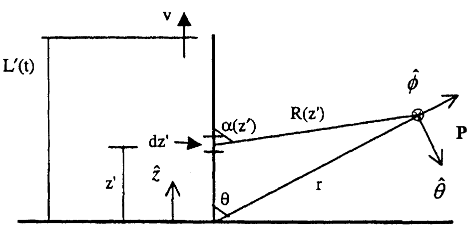

Figure 1. Geometry of the problem

Thus for a return stroke speed equal to the speed of light, the electric and magnetic fields (field derivatives) and the current (current derivative) are shown, by direct calculation starting with the assumed current as a source function, to have identical and invarient waveshapes at all distances and directions.

Discussion

Equations (13) and (19), the fields for speed c, can also be obtained by assuming that a spherical TEM wave propagates at speed c between two concentric conducting conical surfaces with common apexes and solving for the properties of the fields in that waveguide (e.g., Schelkunoff, 1952), that result being applicable to our case of a vertical lightning channel above an infinite ground plane in the limit that the inside cone angle goes to zero and the outside cone angle goes to 180°. Le Vine and Meneghini (1978), using a procedure different from that of this paper, provide an exact expression for the fields due to a current wave travelling at speed c along a filament of finite length. While their expressions (11a) and (11b) do not readily predict identical waveshapes for current, electric field, and magnetic field, it is possible to obtain the same expressions derived in this paper by the approach of Le Vine and Meneghini (1978) if one relaxes the assumption of finite length and considers the field from the current in the whole lightning channel, a step they did not take.

Leteinturier et al. (1990) reported that their electric field derivative and current derivative waveforms measured 50 m from triggered lightning had similar waveshape and that the peak amplitudes were linearly proportional. From the examples presented, the waveform similarity exists for about 100 ns. Uman et al. (2000) also found similar measured electric field and current derivative waveforms 10 to 30 m from triggered lightning for a time of about 100 ns. Thus, we cannot rule out the possibility, assuming the transmission line model is valid at early times in the triggered lightning return stroke, that the return stroke speed during the earliest 100 ns or so could be the speed of light, although there could well be other explanations for the observations. Optical measurements of return stroke speed are not available during the initial 100 ns or so of triggered or natural return strokes, although Wang et al. (1999) used a photoelectric system with a time resolution of 100 ns and an effective spatial resolution of 30 m to show that the average speed in the bottom 60 m of each of two triggered-lightning return strokes was near one-third and near one-half, respectively, of the speed of light, with the speed decreasing with height above 60 m.

Using the radiation or far field approximation (see later in this section), Krider (1992) showed that for the transmission line model with speed v, with the presence of ground taken into account, the radiation field component at an angle θ from the vertical is proportional to the factor $\frac{v}{c}\sin\theta \Big/ \left(1-\frac{v^2}{c^2}\cos^2\theta\right)$. Krider (1992) further showed that for speeds less than about 2.1×10^8 ms^{-1} the maximum field is radiated along the ground surface ($\theta = 90°$), and for greater speeds the maximum field occurs for an angle between 0 and 90°, closer to 0° for higher speeds. When $v = c$, the proportional factor given above becomes, $\sin\theta/1-\cos^2\theta = 1/\sin\theta$ the same factor found in Eqs. (13) and (19) of this manuscript. Note however that Eqs. (13) and (19) give the TOTAL field for the case $v = c$ at ANY distance and do not involve any radiation or far-field approximations.

The previous discussion implies that lightning processes for which the propagation speed may be near c (e.g., perhaps the first 100 ns or so of triggered and the equivalent subsequent natural return strokes from evidence presented here and an undetermined time for first strokes in natural negative and positive flashes) could potentially account for the relatively high fields, higher than observed at ground level, necessary to model "elves," as first suggested by Krider (1994). For example, first return stroke peak fields above 20

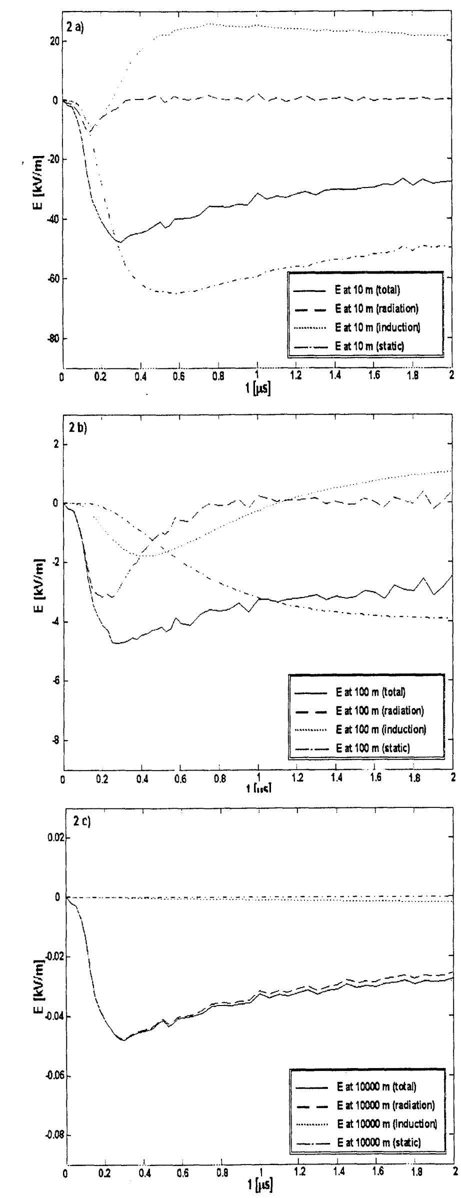

Figure 2. Calculated electric field and its components at $\theta = 90°$ from the transmission line model with $v = c$ and a typical subsequent stroke current waveshape from triggered lightning (S9934-6) at (2 a) 10 m, (2 b) 100 m, and (2 c) 10 km. The small differences in the total field waveshapes at the three distances are due to computational errors.

to 50 V m^{-1} are required by Rowland et al. (1995) in the lower ionosphere at 100 km to produce model elves, whereas the measured mean peak fields at ground level at 100 km are generally 5 to 8 V m^{-1} for negative first strokes, lower for negative subsequent strokes, and about twice that value for positive first strokes, as given by Uman (1987) in Table 7.1 and Table 11.2 and by Rakov and Uman (1990). Note that return stroke electric and magnetic field peaks occur in about 1 μs or less for subsequent strokes and in more than 1 μs for first strokes although roughly half of the first stroke electric field transition to peak can occur in less than 1 μs (e.g., Fisher et al., 1993; Krider et al., 1996; Weidman and Krider, 1980; Willett et al., 1998). Thus, since observed first and subsequent return stroke speeds on a microsecond scale are typically c/3 to c/2 (e.g., Idone and Orville 1982), it follows that at the time of the electric and magnetic field peaks the maximum fields will likely be radiated along the Earth's surface and not in the upward direction.

The commonly-used analytical expression for the vertical electric field intensity at ground level (Figure 1) is (Uman et al., 1975)

$$E(r,t) = \frac{1}{2\pi\varepsilon_o}\int_0^{L'(t)}\left(\frac{(2-3\sin^2\alpha(z'))}{R^3(z')}\int_{t_b(z')}^{t} i(z',\tau - R(z')/c)d\tau + \frac{(2-3\sin^2\alpha(z'))}{cR^2(z')}i(z',t-R(z')/c) - \frac{\sin^2\alpha(z')}{c^2R(z')}\frac{\partial i(z',t-R(z')/c)}{\partial t}\right)dz', \quad (20)$$

where the integral limits are discussed, for example, by *Thottappillil et al.* [1997]. The electric field is composed of, from left to right in Eq. (21), three components defined as the electrostatic field, the induction field, and the radiation field (Uman et al., 1975). In Figure 2 we show a transmission line calculation for return stroke speed c of the components of the electric field intensity from Eq. (20) for a causative triggered lightning return stroke current that rises to a peak of 8.5 kA in about 200 ns and falls to one-half of peak value in about 3 μs. Calculations are given for three distances, 10 m, 100 m, and 10 km. While the total electric field always has an identical waveshape to that of the current, as expected from Eqs. (13) and (19), the proportion of the field components that comprise those waveshapes changes markedly with distance. At 10 m, the electrostatic field dominates the total field peak, whereas at 10 km the radiation field dominates the total field for the 2 μs duration of waveform shown. It is worth noting that the classical division of the total electric field into static, induction, and radiation components, as expressed in Eq. (20) is not unique, as shown by Thottappillil and Rakov (2001). That is, there are other formulations of individual terms that sum to the same total field.

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